Wave Equation Moveout

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Introduction

This paper presents normal moveout (NMO) from the viewpoint of wave-equation migration. An advantage to this approach, though it may be more expensive than streamlined methods of NMO in industry, is that it shows the actual physical approximations assumed in standard NMO and points the way to more accurate methods.

It will be shown that standard NMO is a special case of a general, exact moveout procedure. Essentially it is a ray approximation to the exact equations, without filtering or any amplitude correction due to spreading. The amplitude correction for spreading is derived by applying the ray approximation to wave NMO. Further, there exists a class of intermediate procedures between ray methods and the wave method of NMO which promise to be relatively cheap and robust with respect to non-distortion of waveforms in the trace.

Wave-equation Moveout -- Derivation

In this section the equations for performing an "ideal" normal moveout will be derived. First let us define our problem: derive a procedure that takes one trace from a common midpoint gather and transforms it to a moved-out trace. The resultant traces can be vertically summed together to form the zero-offset stacked trace. Our procedure is based on a wave equation, so that it will be exact insofar as the governing equation we are using is exact. It should be quasi-linear, so that there is little resultant nonlinear distortion in waveforms on the trace. Another desirable feature is reversibility. The ability to transform back and forth between the common midpoint gather and the moved-out gather allows filtering in one domain or the other.

To justify using a wave-equation operator to perform the combined operations of moveout and stack on a common midpoint gather, the flat-earth assumption has to be made: beds are horizontal, and velocity stratification is horizontal. In this case the double square root equation simplifies to a migration in offset only. The double square root equation in the (ω, k_h, k_y) domain is (Clayton, SEP-14, p. 25):

$$\frac{\partial q}{\partial z} = \frac{i\omega}{v(z)}\left[\sqrt{1-(H+Y)^2}+\sqrt{1-(H-Y)^2}\right] q \qquad (1)$$

where $H = \frac{vk_h}{2\omega}$ and $Y = \frac{vk_y}{2\omega}$. The spatial wavenumbers of half-offset and midpoint, are, respectively, k_h and k_y. For a flat earth, $k_y = 0$ for any nonzero part of the data field q, so that (1) reduces to

$$\frac{\partial q}{\partial z} = \frac{i\omega}{v(z)} 2\sqrt{1-H^2}\, q \qquad (2)$$

Now change coordinates from z to migrated two-way travel time τ,

$$\tau = 2\int_0^z \frac{dz}{v(z)}, \qquad \frac{d\tau}{dz} = \frac{2}{v(z)}$$

so that

$$\frac{\partial q}{\partial \tau} = i\omega\sqrt{1-H^2}\, q\ . \qquad (3)$$

This is the equation governing downward continuation of a common-depth point gather that combines the operations of moveout and stacking. We can talk about CDP gathers here rather than CMP gathers since the flat-earth assumption now prevails, and gathers do possess a common depth point.

Before continuing, note that the sign conventions used in this paper are:

$$q(\omega) = \int_{-\infty}^{+\infty} dt\ e^{-i\omega t} q(t),$$

$$q(k_h) = \int_{-\infty}^{+\infty} dh\ e^{ik_h h} q(h).$$

Migration involves a waveform either travelling forward in time and backward in z, or backward in time and forward in z (e.g. $\delta(t + z/v)$). For a delta waveform,

$$q(\omega) = \int dt\ e^{-i\omega t}\delta(t + \frac{z}{v}) = e^{i\omega\frac{z}{v}}.$$

This is the solution of the simple downward continuation equation

$$\frac{dq}{dz} = i\omega\frac{z}{v}q$$

so that the sign of the right hand side of (3) agrees with the sign here (positive $i\omega$).

Now the moveout and stack operators can be separated by considering the application of (3) to each separate trace of the CDP gather. Data on other traces are assumed to be zero. Migration creates a two-dimensional field out of this trace, but only the zero-offset trace of this field is retained. After performing this operation of moveout to each trace (converting it to a zero-offset trace) all traces may be summed together to give the migrated field q(τ, t=0, h=0), since the entire operation done by (3) is linear. The moveout traces have thus been stacked.

Let $q_h(t)$ be a CDP trace at half-offset h. The wavefield to be migrated looks like $q(h', t) = \delta(h'-h)q_h(t)$, or,

$$q(k_h, \omega) = \int_{-\infty}^{+\infty} dt\ e^{-i\omega t} q_h(t) \int_{-\infty}^{+\infty} dh'\ \delta(h'-h)\ e^{ik_h h'}$$

$$= q_h(\omega)\ e^{ik_h h}$$

The migrated wavefield, via (3), is simply

$$q(k_h, \omega, \tau) = e^{i\omega\tau\sqrt{1-H^2}}\ q(k_h, \omega)$$

The desired zero-offset trace is found at t=0 and h=0:

$$q(\tau) \equiv q(\tau, h=0, t=0) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} dk_h \int_{-\infty}^{+\infty} d\omega\ e^{i\omega\tau\sqrt{1-H^2}}\ q(k_h, \omega)$$

or,

$$q(\tau) = \frac{1}{4\pi^2}\int\!\!\int_{-\infty}^{+\infty} dk_h\, d\omega\ e^{i\omega\tau\sqrt{1-H^2}}\ q(\omega)\ e^{ik_h h} \qquad (4)$$

where $H = \frac{vk_h}{2\omega}$.

Now introduce the ray parameter $p = \frac{k_h}{2\omega}$. It is the physical ray parameter which corresponds to the emergence angle of the ray: $p = \frac{\sin\vartheta}{v}$. The factor of one half is present because our coordinate system is in two-way time τ versus half-offset h. The true p is half the slope seen on the time-distance curve in our coordinate frame:

$$p = \frac{1}{2}\frac{dt}{dh}, \qquad k_h = 2\omega p.$$

The differential of the double integral (4), by examination of the Jacobian, is $dk_h\, d\omega = 2|\omega|\, dp\, d\omega$. Then (4) becomes

$$q(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} |\omega|\, d\omega\; q(\omega)\; e^{i\omega(\tau\sqrt{1-v^2p^2} + 2ph)} \tag{5}$$

This is our basic equation. The procedure implied by (5) is the following:

a) First, rho filter. This term, borrowed from the field of tomography, means apply the filter $|\omega|$ to $q(\omega)$. If $\tilde{q}(\omega) \equiv |\omega|\, q(\omega)$, then

$$q(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} d\omega\; \tilde{q}(\omega)\; e^{i\omega(\tau\sqrt{1-v^2p^2} + 2ph)},$$

$$q(\tau) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dp\; \tilde{q}(t = \tau\sqrt{1 - v^2p^2} + 2ph). \tag{5a}$$

b) Stretch each individual p-trace:

$$\overline{q}(\tau) \equiv \tilde{q}(\tau\sqrt{1 - v^2p^2}).$$

Notice this is an elliptical moveout rather than hyperbolic moveout. Equation (5a) becomes:

$$q(\tau) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dp\; \overline{q}(\tau + 2ph). \tag{5b}$$

c) Finally, note (5b) is a slant stack over p with slope 2h. Without any significant loss of accuracy, the p values to be stacked only need to range over the propagating window $-1/v \leq p \leq 1/v$. Outside this range, the square root of (5) becomes imaginary, where waves corresponding to these values of p are evanescent.

Once all the offset traces q_h are moved out, the procedure can be reversed. An inverse slant stack over h now has to be performed with slopes -2p. The p-gather of equation (5a) is re-obtained when rho filtering is subsequently applied (Thorson, SEP-14, p. 81).

A theoretical advantage to wave equation NMO is that it is non-distorting. Downward continuation is an all-pass filter in the direction τ, so that the operations described above will not change the color of each offset trace. Rho filtering can be thought of as counteracting the smoothing effect of the integration in (5b), so that the resultant filtering performed on an offset trace is simply $\frac{|\omega|}{i\omega}$, which is simply a Hilbert transform. There is a subsequent disadvantage here. The amount of computation required to produce one moved-out trace is n_p times as much as the standard process, where n_p is the number of p-traces used in the sum of (5b). If not enough p-traces are slant-stacked together, poor results are obtained at large cost. The following sections of this paper describe some solutions to this numerical problem by examining "better" approximations to the p-integral in (5b). An obvious idea is to apply a stationary phase approximation to perform integration over p -- essentially integrating around a single p value corresponding to the ray arriving at half-offset h. In the next section it is shown that this is equivalent to a dressed-up version of standard NMO.

Stationary Phase Evaluation of (5)

In (5), the complete double integral representing the moved out trace $q(\tau)$ is given by

$$q(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} q(\omega)\ |\omega|\ J(\omega,h,\tau)\, d\omega \tag{6}$$

where

$$J(\omega,h,\tau) = \int_{-\infty}^{+\infty} e^{i\omega\Theta(p,h,\tau)} dp \tag{6a}$$

and

$$\Theta(p,h,\tau) = 2ph + \sqrt{1-p^2v^2}\,\tau$$

To evaluate (6), we do the p integral first by the method of stationary phase. This method is used especially for integrals which have the same form as the p integral, denoted by J. The stationary phase method is valid when the parameter multiplying the phase function approaches infinity. In our case the large parameter is the frequency ω. Thus, the stationary phase method, applied to the p integral or p-stack, is a high frequency approximation. When this approximation is integrated over frequency to obtain q(τ), the result is a ray or geometrical approximation to the complete double integral in (6).

In order to evaluate J, we find the point or points where the phase Θ is stationary, since everywhere else the integrand is oscillating so rapidly that it cancels itself out. For the given offset, the phase is stationary when $\frac{d\Theta}{dp} = 0$. For the phase function defined above, this occurs when

$$p = \frac{2h}{v\sqrt{4h^2 + v^2\tau^2}}$$

Before actually applying the method, we shall need the values of Θ and its second derivative at the stationary point, p_o. These values are given as

$$\Theta(p_o) = \frac{\sqrt{4h^2 + v^2\tau^2}}{v}$$

and

$$\left.\frac{d^2\Theta}{dp^2}\right|_{p=p_o} = -\frac{(4h^2 + v^2\tau^2)^{\frac{3}{2}}}{v\tau^2}$$

Utilizing the values defined above, we expand the phase Θ about the stationary point, so that

$$\Theta(p) = \frac{\sqrt{4h^2 + v^2\tau^2}}{v} - \frac{(4h^2 + v^2\tau^2)^{\frac{3}{2}}}{2v\tau^2}(p - p_o)^2 \tag{7}$$

Then substitution of (7) into (6a) results in

$$J(\omega,h,\tau) = e^{\frac{i\omega}{v}\sqrt{4h^2 + v^2\tau^2}} \int_{-\infty}^{+\infty} e^{\frac{-i\omega}{2v\tau^2}(4h^2 + v^2\tau^2)^{\frac{3}{2}}(p - p_o)^2} dp \tag{8}$$

By contour integration, the integral in (8) can be easily evaluated. Therefore,

$$J(\omega,h,\tau) = \sqrt{\frac{2\pi v\tau^2}{|\omega|(4h^2 + \tau^2v^2)^{3/2}}}\; e^{\frac{i\omega}{v}\sqrt{4h^2 + v^2\tau^2} - \frac{i\pi}{4}sgn(\omega)} \tag{9}$$

Substitution of (9) into (6) yields the moved out trace $q(\tau)$

$$q(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} q(\omega)|\omega| \sqrt{\frac{2\pi v\tau^2}{|\omega|(4h^2 + \tau^2v^2)^{3/2}}}\; e^{\frac{i\omega}{v}\sqrt{4h^2 + v^2\tau^2} - \frac{i\pi}{4}sgn(\omega)} d\omega \tag{10}$$

The integral in (10) can be evaluated by inspection so that

$$q(\tau) = \frac{1}{\sqrt{2\pi^3}} \sqrt{\frac{v\tau^2}{(4h^2 + v^2\tau^2)^{3/2}}}\; \hat{q}(\sqrt{4h^2 + v^2\tau^2}) \tag{11}$$

where

$$\hat{q}(t) = \int_{-\infty}^{+\infty} \sqrt{|\omega|}\, e^{-\frac{i\pi}{4} sgn(\omega) + i\omega t} q(\omega) d\omega$$

A cursory glance at (11) demonstrates that the recipe for a ray-type NMO has three steps. First, the trace is stretched by the normal hyperbolic factor, $\sqrt{4h^2 + v^2\tau^2}$. Then, the two-sided filter $\sqrt{-i\,\omega}$ is applied. Finally, at each offset, the trace is scaled by an offset and time dependent amplitude factor. As is the case with NMO done the conventional way, there is some distortion as events are move-out corrected.

The "Disk-Ray" Approach to Wave Equation Moveout

Improvements in the efficiency of using equation (5) revolve about the solution of the integral

$$J \equiv \int dp\ e^{i\omega\Theta(p)} \tag{12}$$

where for our particular case $\Theta(p) = \sqrt{1 - v^2p^2}\tau + 2ph$. In the last section, a parabola was fit to the phase function $\Theta(p)$ at the point of stationary phase, which incidentally was the maximum phase. The result was normal moveout with extra filtering and amplitude corrections applied. In this section, integral (12) is approximated by fitting straight line segments to the phase function $\Theta(p)$ (figure 1). The objective is to perform a good moveout with a sum over relatively few p values. The method is reminiscent of the disk-ray modeling of Wiggins (1976).

Suppose we divide the path of the phase function $\Theta(p)$ in the propagating region into pieces and approximate each piece by a straight line segment (curve Q of figure 1). Then

$$J = \int dp\ e^{i\omega\Theta(p)} = \sum_{i=1}^{n} \int_{p_{i-1}}^{p_i} dp\ e^{i\omega\Theta(p)}$$

$$J \approx \sum_{i=1}^{n} \int_{p_{i-1}}^{p_i} dp\ \exp i\omega\left[\Theta_{i-1} + \frac{\Delta\Theta_i}{\Delta p_i}(p - p_{i-1})\right]$$

where

$$\Delta\Theta_i \equiv \Theta_i - \Theta_{i-1},$$

$$\Delta p_i \equiv p_i - p_{i-1},$$

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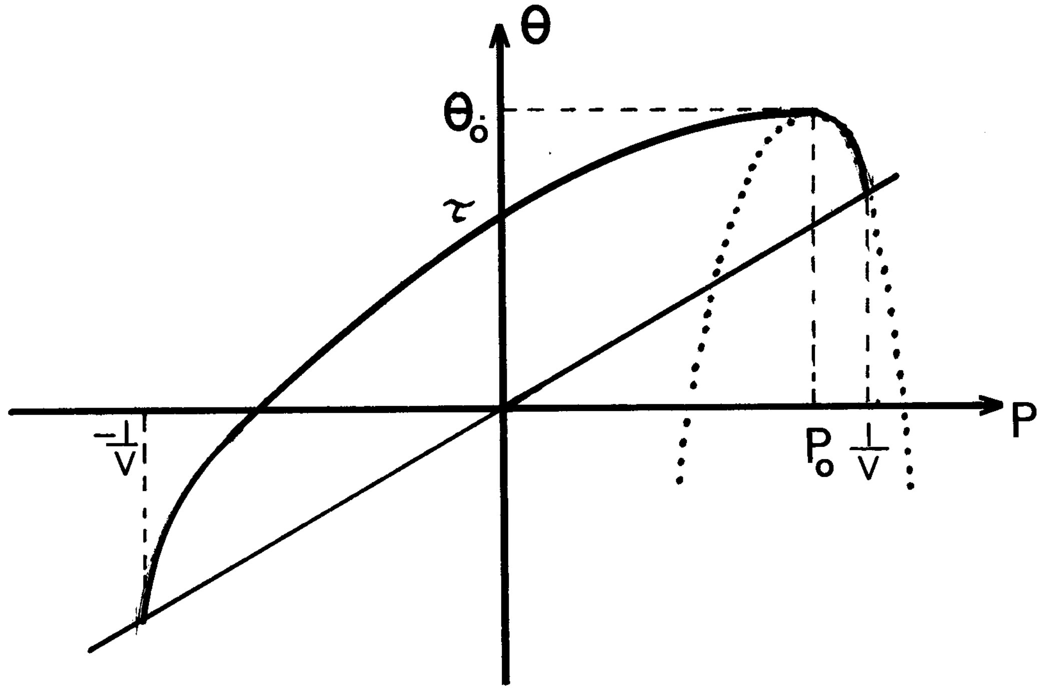

FIG. 1. The phase function $\Theta(p)$. It is a slanted ellipse with vertical intercept τ and range $-1/v \leqq p \leqq 1/v$. This represents propagating values of p. In the stationary phase case $\Theta(p)$ is approximated by the parabola P fitting it at it's maximum value. $\Theta(p)$ can also be linearly interpolated at the points p_i: curve Q. In any case the main contribution to the integral (12) occurs near the maximum $p = p_0$ where the curve goes flat.

$$\Theta_i \equiv \Theta(p_i).$$

Δp can vary, since the p_i's are free to be chosen. With the above approximations, the integrals inside the sum are easy to evaluate:

$$\int_{p_{i-1}}^{p_i} dp \, \exp i\omega\left[\Theta_{i-1} + \frac{\Delta\Theta_i}{\Delta p_i}(p - p_{i-1})\right]$$

$$= \exp i\omega\left[\Theta_{i-1} - \frac{\Delta\Theta_i}{\Delta p_i} p_{i-1}\right] \frac{\Delta p_i}{i\omega\Delta\Theta_i}\left[\exp i\omega\frac{\Delta\Theta_i}{\Delta p_i} p_i - \exp i\omega\frac{\Delta\Theta_i}{\Delta p_i} p_{i-1}\right]$$

$$= \frac{\Delta p_i}{i\omega\Delta\Theta_i}\left[e^{i\omega\Theta_i} - e^{i\omega\Theta_{i-1}}\right] \qquad (13)$$

Therefore the integral (12) is simply the sum of terms (13). Now place this result into equation (5).

$$q(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty}\frac{|\omega|}{i\omega}d\omega\, q(\omega)\sum_{i=1}^{n}\frac{\Delta p_i}{\Delta\Theta_i}\left[e^{i\omega\Theta_i} - e^{i\omega\Theta_{i-1}}\right]$$

In this case, the filter to apply to the initial traces q(t) is a Hilbert transform: $\frac{|\omega|}{i\omega} = -isgn(\omega)$. Now let

$$\tilde{q}(t) \equiv -Hi(q(t)) = F.T.^{-1}\left\{-isgn(\omega)q(\omega)\right\}.$$

Then

$$q(\tau) = \frac{1}{\pi}\sum_{i=1}^{n}\frac{\Delta p_i}{\Delta\Theta_i}\left[\tilde{q}(\Theta_i) - \tilde{q}(\Theta_{i-1})\right] \quad (14)$$

Reorganizing this gives the desired result

$$q(\tau) = \frac{1}{\pi}\sum_{i=1}^{n}\left[\frac{\Delta p_i}{\Delta\Theta_i} - \frac{\Delta p_{i+1}}{\Delta\Theta_{i+1}}\right]\tilde{q}(\Theta_i) \quad (15)$$

(where we assume $\Delta p_{n+1} = 0$). One can immediately see that the contribution to the sum occurs where the curvature of $\Theta(p)$ is large. Very little is contributed to the integral where $\Theta(p)$ is flat. This provides a guideline for picking the nodes p_i: cluster them where there is curvature and spread them out on the flat parts. To be safe, one p_i should always be the principle ray parameter p_0 of the previous section, so that $\Delta\Theta_i$ in (14) cannot by chance drop to zero. It is interesting to note that in the limit, (14) approaches

$$q(\tau) = \frac{1}{\pi}\int_{-1/v}^{+1/v} dp \left[\frac{d\tilde{q}}{d\Theta}\right]$$

which corresponds to equation (5).

For τ-variable velocity, Θ has the form

$$\Theta(p) = \int d\tau\,\sqrt{1 - v^2(\tau)p^2} + 2ph.$$

The preceding analysis follows through in the same way. To obtain values of $\Theta(p)$ from this equation, table lookups can be implemented, when the RMS velocity curve is known.

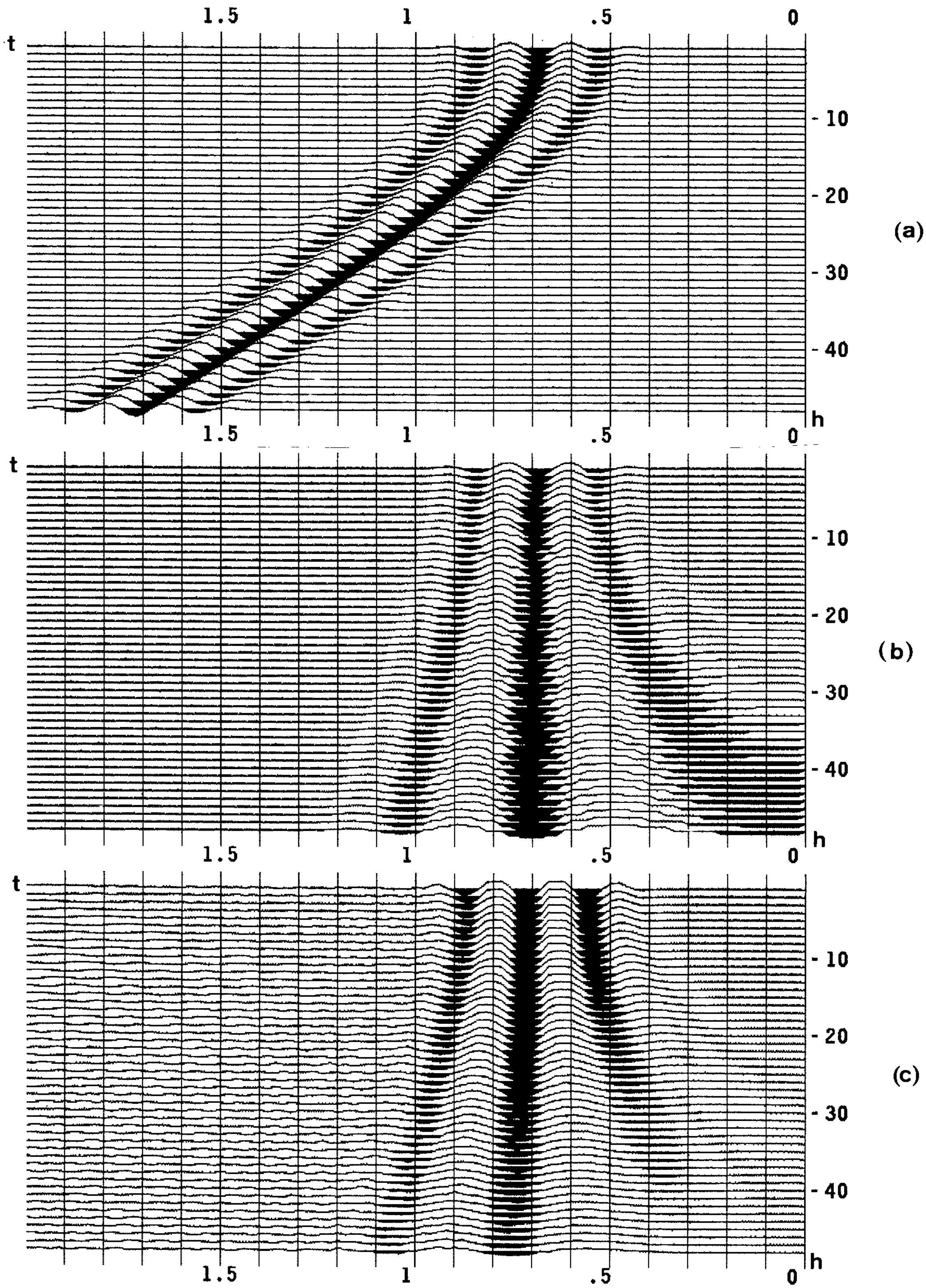

FIG. 2. (a): Artificial common-midpoint gather generated with a constant velocity of 1500 m/sec and a constant waveform. The parameters chosen for this model were: 0.02 sec sample interval, 25 m trace interval, 48 traces and 100 samples per trace. Zero offset travel-time of the event is 0.66 seconds. (b): Normal moveout correction of (a) (at a velocity of 1500 m/sec). (c): Wave equation moveout correction of (a) at the correct velocity.

Example

Figure 2(a) shows a trial common midpoint gather made from a constant velocity model of 1500 m/sec. It is a simple waveform superimposed on a hyperbolic trajectory. Therefore, when whis event is moveout-corrected, the flanks of the waveform will be over- or under-corrected because of the apparent different velocity of the flanks.

With the correct velocity, a crude standard NMO is applied to figure 2(a). The results are shown in figure 2(b). For comparison, the results of applying equation (5) to the gather in figure 2(a) are shown in figure 2(c). Forty-eight p-values were used in the p-stack. It gives virtually the same response as NMO, except for an obvious decay in amplitude at larger offsets. This is due to the fact that no amplitude adjustment was done for the case of standard NMO.

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