WAVE EQUATIONS FOR SNELL-WAVE MULTIPLES

[condensed from SEP-15, p. 191-202 and SEP-20, p. 57-72]

In the last two sections we began with a simple recursive relation to account for multiple reflections in a perfectly layered earth. We worked up to a relation for a layered earth where reflection coefficients could be varied laterally. These relations were

$$u_t = \sum_{z=1}^{t} c_z \, d_{t-z} \tag{0a}$$

$$u_t^x = \sum_{z=1}^{t} c_z^x \, d_{t-z}^{x-z} \tag{0b}$$

It was asserted that (0b) could be used for dips up to 5 degrees, but dip was never considered explicitly. Now we will develop a wave-equation method good for multiples traveling at angles of 25 degrees or more. It specializes at small dip to the simple intuitive relation (0b). Additionally we will make a significant extension of (0b) to incorporate depth variation in velocity.

Traveltime Depth

In geophysical data processing it is advisable to avoid, if possible, use of information which is not well known. The sensitivity of each process to erroneous inputs is learned by both experience and organized analysis. For example, the velocity inside the earth is rarely well known and the degree of uncertainty affects different processes differently. Often the output of seismic analysis is a seismic section which is a picture of the earth as seen in some (x,z)-plane. To reduce sensitivity to velocity, the picture is almost universally presented in (x,τ)-space where τ is a two-way traveltime variable related to depth by

$$\tau = 2 \int_0^z \frac{dz}{v(z)} \tag{1a}$$

Wave equations may look a little strange in terms of the variable τ instead of the variable z but this transformation is particularly important in the analysis of multiple reflections. It is difficult to suppress multiples and we often fail. By plotting the results in terms of a time variable τ instead of a depth variable z we may be able to recognize residual multiple reflections by their familiar timing relationships.

Normal-Incidence Retarded Coordinates

Normal-incidence retarded time t' is defined by

$$t' \quad = \quad t - \int \frac{dz}{v(z)} \tag{1b}$$

The downgoing wave d can be expressed in the usual (z,t)-coordinates or in retarded coordinates as $d'(\tau,t')$. The downgoing wave may have some arbitrary waveform $f(t - t_0)$ where t_0 is the delay expressed by the integral in (1b). Taking the wave to be going straight down, we have

$$d(z,t) \quad = \quad f\left[t - \int \frac{dz}{v(z)}\right] \quad = \quad d'(\tau,t') \quad = \quad f(t')\ \mathrm{const}(\tau) \tag{2}$$

This equation is the solution to the differential equation

$$\frac{\partial}{\partial \tau} d' \quad = \quad 0 \tag{3}$$

This restates our initial assumption that the downgoing wave is depth-invariant. We also made this assumption when we developed equation (0a).

We have another retarded time t'' for upcoming waves obtained by changing the sign of the depth axis in the downgoing definition (1b):

$$t'' \quad = \quad t + \int \frac{dz}{v(z)} \tag{4}$$

Subtracting (1b) from (4) and using (1a) gives

$$t'' - t' \quad = \quad 2 \int \frac{dz}{v(z)} \quad = \quad \tau \tag{5}$$

Waves or reflection coefficients may be expressed mathematically in any coordinates, so we may take our choice from

$$d(z,t) \;=\; d'(\tau,t') \;=\; d''(\tau,t'') \tag{6a}$$

$$u(z,t) \;=\; u'(\tau,t') \;=\; u''(\tau,t'') \tag{6b}$$

$$c(z) \;=\; c'(\tau) \;=\; c''(\tau) \tag{6c}$$

The differential equation

$$\frac{\partial}{\partial\tau} u'' \;=\; c\,d \;=\; c''d'' \tag{7a}$$

has the solution

$$u''(\tau,t'') \;=\; \int d\tau\; c''(\tau)\; d''(\tau,t'') \tag{7b}$$

Changing from d'' to d' with (6a) and using (5) to get all *independent* variables into double prime space we get

$$u''(\tau,t'') \;=\; \int d\tau\; c''(\tau)\; d'(z'',t'' - \tau) \tag{8}$$

Equation (8) has the same meaning as equation (0a). A superficial difference is that the depth axis is discretized in equation (0a).

In summary, the differential equations which we seek in order to control the wave fields will specialize at vertical incidence, in the absence of diffraction, to the pair

$$\frac{\partial}{\partial\tau} d' \;=\; 0 \tag{3)or(9a}$$

$$\frac{\partial}{\partial\tau} u'' \;=\; c''(\tau)\; d'(\tau,t'' - \tau) \tag{9b}$$

The equations used by Riley and Claerbout (1976) were just these, along with

∂_{xx}^{t} diffraction terms. Now we seek the equations of the slanted system which will relate to (0b).

Slanted-Incidence Retarded Coordinates

We begin by defining a coordinate frame which turns out to be the natural coordinate frame for use in computation of downgoing waves. This coordinate frame is simplest conceptually when velocity v is constant. But in operation it is no more difficult with depth-variable velocity v(z). So we will state it both ways:

$$t' \;=\; t - x\,\frac{\sin\theta}{v} - z\,\frac{\cos\theta}{v} \;=\; t - px - \int \frac{\cos\theta}{v}\,dz \qquad (10a)$$

$$x' \;=\; x - z\tan\theta \;=\; x - \int \tan\theta\,dz \qquad (10b)$$

$$\tau \;=\; 2z\,\frac{\cos\theta}{v} \;=\; 2\int \frac{\cos\theta}{v}\,dz \qquad (10c)$$

The last equation defines two-way traveltime depth for a wave at angle θ. Note that the time-to-depth conversion is not the downward speed $v\cos\theta$ of the tip of a ray but the speed $v/\cos\theta$ of a wavefront as seen in a borehole. The definition of x' by (10b) is such that if you force x' to be constant then the resulting constraint $dx = dz\tan\theta$ will keep you on a ray. The definition of t' by (10a) is such that fixing both x' and t' implies that $dz = v\,dt\cos\theta$, which means you move downward at the speed of a ray.

Consider first an impulsive plane wave traveling downward into the earth at an angle θ from the ray to the vertical. A mathematical expression for this wave can be given in Cartesian coordinates or in the natural coordinates as

$$d \;=\; \delta\left(t - x\,\frac{\sin\theta}{v} - z\,\frac{\cos\theta}{v}\right) \;=\; d' \;=\; \delta(t')\,\mathrm{const}(x')\,\mathrm{const}(\tau) \qquad (11)$$

To verify the direction of this wave we may observe the location of the wavefront by setting the argument of the delta function to zero. At t=0 the wavefront is thus located at $z = -x\tan\theta$. Observing other t values it may be noted that the wave moves at speed v in the direction $(dx,dz) = (\sin\theta, \cos\theta)$. It is important to note the form of this downgoing wave at the earth's surface z=0, namely $\delta[t - (x/v)\sin\theta]$. This is the behavior of a Snell-wave source.

Next consider the downgoing wave that will be seen in a stratified medium $[c = c(z),\ v = \mathrm{const}(z)]$, including the multiple reflections. In such a one-dimensionally variable medium we need a 1-D function, say f(arg), to describe the seismograms. We may assert that within our usual approximations on downgoing waves an appropriate mathematical expression is

$$d = f\left(t - x\frac{\sin\theta}{v} - z\frac{\cos\theta}{v}\right) = d' = f(t')\ \mathrm{const}(x')\ \mathrm{const}(\tau) \qquad (12)$$

Observe that at the surface z=0 the downgoing wave (which is related to the observed wave u by u = s-d) reduces to the form $f[t - (x/v)\sin\theta]$. Deep echos may be sensed in either of two ways: (1) stay at constant x and look at large positive t; or (2) take a snapshot (constant t) of the earth's surface and look at large negative x. In (12) we see that the advantage of the natural coordinates is that a physical function of the three variables (x,z,t) is expressed as a function of the single natural variable t'.

Next consider a situation where there is variation along the x-axis of either the surface source or the reflection coefficients $c = c(x,z)$. Now the downgoing wave is obviously at least a 2-D function, say

$$d = d' = f(t',x')\ \mathrm{const}(\tau) \qquad (13)$$

Why is it that we can assert that the depth-dependence τ in (13) is that of a constant function? The reason goes back to the definitions (13a,b,c). Clearly, expression (13) is a constant if x' and t' are constants. The constancy of x' forces us to stay on a ray. The constancy of t' forces us to move downward at the speed of a ray. So the wave d should not appear to change if we are moving with it, meaning that (13) should be a constant function of τ. Now we may write a partial-differential equation for the downgoing wave

$$\frac{\partial}{\partial\tau} d' = 0 \qquad (14)$$

where it is understood that the partial differentiation is done at constant x' (on a ray) and constant t'.

Obviously an appropriate change of sign for the z-dependent terms in equation (10) gives us a natural set of coordinates for computing ***upcoming*** waves. This change is

$$t'' = t - px + \int \frac{\cos\theta}{v} dz \tag{15a}$$

$$x'' = x + \int \tan\theta \, dz \tag{15b}$$

$$\tau = 2 \int \frac{\cos\theta}{v} dz \tag{15c}$$

In (15c) both z and τ have had signs changed, so nothing changes.

Since up- and downgoing waves will get coupled at the reflectors we will need to relate the two different sets of natural coordinates. Subtracting (15) from (10) we get

$$t' - t'' = 2 \int \frac{\cos\theta}{v} dz = \tau \tag{16a}$$

$$\begin{aligned} x' - x'' &= 2 \int \tan\theta \, dz \\ &= 2 \int \tan\theta \frac{dz}{d\tau} d\tau \\ &= 2p \int_0^{\tau} \frac{v^2}{\cos^2\theta} d\tau \triangleq \chi(\tau) \end{aligned} \tag{16b}$$

The latter equation defines Morley's shift function, a lateral shift $\chi(\tau)$ between x' and x'' which will occur explicitly in the processing equations.

Coupling of Up- and Downgoing Wave Fields

We seek an equation like (0b), (8), or (9b) to sum up the contribution to the upcoming wave field from its source, namely the products of the reflection-coefficient field with the downgoing wave field. If for the moment we focus our attention on layered media, then a rigorously correct equation may be derived from Claerbout's (1976) equation 10-5-1 along with the weak-reflector assumption $|U| << |D|$. It is

$$\frac{\partial}{\partial z} u = \frac{\cos \phi}{v} \frac{\partial}{\partial t} u + \frac{1}{2} \frac{Y_z}{Y} d \tag{17}$$

where $\cos \phi$ is the cosine of the propagation angle. This angle may be found by square-root expansions of partial-differential operators as in Claerbout (1976) or it may be found in the 2-D Fourier-transform space of temporal frequency and horizontal spatial frequency. The right-hand term of equation (17) contains the product of the downgoing wave d and a reflection-coefficient-like object Y_z/Y. Study of this matter will show that Y also involves the cosine of the angle. Because of our practical intent and the experimentalist philosophy of adding new complications only one at a time, we will choose to ignore the angle-dependence of reflection coefficients. Equation (17) of course also assumes horizontal bedding, thereby introducing errors of the same order. In light of all these considerations the coupling equation will be taken to be

$$\left(\frac{\partial}{\partial z} - \frac{\cos \phi}{v} \frac{\partial}{\partial t}\right) u = c\, d \tag{18}$$

Those experienced in wave-equation methods will note directly that if $\theta = \phi$ the left side of (18) reduces to $\partial U'/\partial \tau$. This may be called the 5-degree dip equation. Higher square-root expansions give the so-called 15-degree and 45-degree equations. Keeping only the 5-degree equation we have

$$\frac{\partial}{\partial \tau} u'' = c\, d = c''d'' = c''d' \tag{19}$$

As usual we want to compute u in its natural representation $u''(x'',\tau,t'')$ and d in its natural representation $d'(x',\tau,t')$.

$$\frac{\partial}{\partial\tau} U'' = c''(x'',\tau)\ d'(x',\tau,t')$$

Using (16) to get all independent variables in terms of the double prime frame we get

$$\frac{\partial}{\partial\tau} U'' = c''(x'',\tau)\ d'[x''-\chi(\tau),\tau,t''-\tau] \tag{20}$$

This equation and equation (0b) are saying much the same thing. A superficial difference is that (20) is a differential equation with respect to traveltime depth τ whereas (0b) is summed (like integration) over depth z. A more basic difference is that the lateral shift term $\chi(\tau)$ may be a function of the depth-variable velocity, but the corresponding z superscript on d in the above equation can have validity only for a particular mesh ratio in a constant-velocity medium. Glancing back at equation (16b), which defined χ, we see that it goes as the time average velocity squared. So in practice the improvement should be substantial.

We have reached sufficient height that we get our first overall view of the territory. The intuitive modeling and processing equation (0b) section is effectively a special case of a wave-extrapolation equation (17). From here there are many directions in which we could move to process data with more or less accuracy, the price for accuracy being complexity. There are many directions to go in getting different kinds of accuracy, and the way you go will depend in large measure on your dataset. The most obvious direction is to incorporate the usual diffraction terms which extend the range of angular validity.

Statement of Diffraction Equations

Observing all the conventions of equation (19) but for simplicity omitting all the primes, we get (I hope)

$$D = S - U \qquad \text{at} \quad \tau = 0 \tag{21a}$$

$$\frac{\partial}{\partial\tau} D = \frac{v^2}{4\cos^4\theta}\, \partial^t_{xx}\, D \tag{21b}$$

$$\frac{\partial}{\partial \tau} U = \frac{-v^2}{4 \cos^4\theta} \partial^t_{xx} U + c(x,\tau) \; D[x-\chi(\tau),\tau,t-\tau] \qquad (21c)$$

These equations may be implemented in much the same way as in Riley and Claerbout (1976) or Claerbout (1976). What is new is the lateral shifting terms.

Synthetic Multiples on a Bright-Spot Model

Figure 1 shows a bright-spot model and various synthetic seismic sections computed by Raul Estevez. The model is shown in frame a. Slanting is present in frames c and e, whereas diffraction is present in frames d and e.

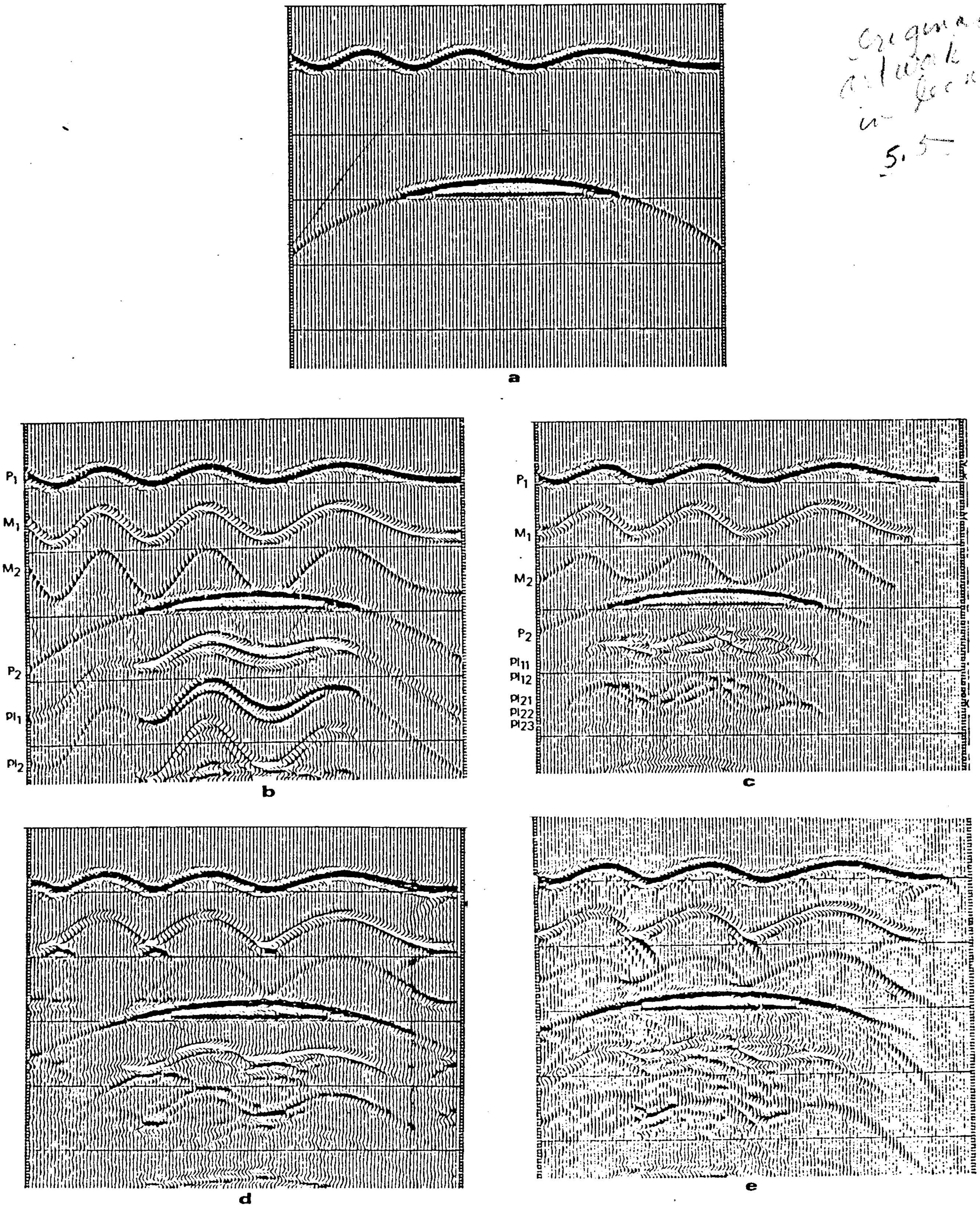
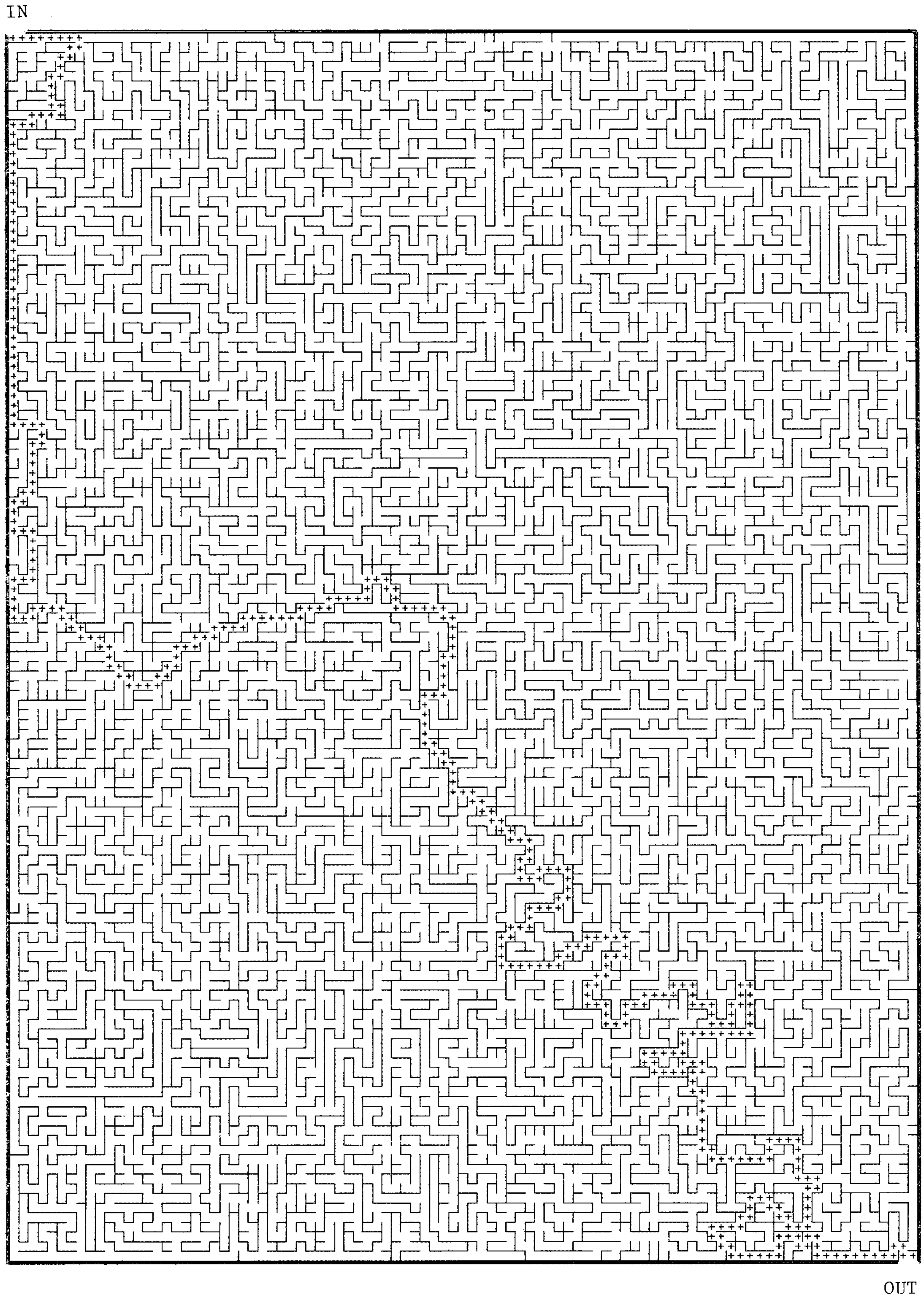


FIG. 1. Bright-spot model and synthetic seismic sections.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