IMPEDANCE AND WAVE-FIELD EXTRAPOLATION

[*adapted from SEP-16, p. 131-154*]

In describing stable physical processes rarely is much attention given to the stability of the modeling equations. The common feeling is that since the physical process is stable, so must be any correct and reasonably accurate modeling equations. This is so often true that concern with stability, like concern with existence proofs, is frequently regarded as highly academic.

Quite the opposite circumstance applies in geophysical data processing where we are involved with the ***inverse*** of physical modeling. Modeling, the way nature does it, is extrapolation forward in time. Extracting information about the earth's interior from surface measurements is inverse modeling. Such extraction is really extrapolating information in depth. Nature does boundary value problems in depth, not initial value problems, so we can consider ourselves lucky when we are able to extrapolate downward. When a depth extrapolation is stable then we simply cannot determine the information we seek.

Instability commonly arises from one of the following two causes:

1) Mathematical equations may have a unique solution, but there may be a ridiculous sensitivity to data accuracy.

2) Approximations which are reasonable and valid in the frequency range of interest might violate causality outside that range.

In any practical situation there is obviously a great need to know which of the above two situations is applicable. Luckily in seismic imaging we are usually in case (2). To regain stability the main requirement is that we learn some stability analysis and use it. Of all the virtues a computational algorithm can have - stability, accuracy, clarity, generality, speed, modularity, etc. - the most important seems to be stability.

Beware of infinity!

To prove that one equals zero you take an infinite series 1,-1,+1,-1,+1,$\cdots$, group the terms in two different ways, and add them thus:

$$(1-1) + (1-1) + (1-1) + \cdots = 1 + (-1+1) + (-1+1) + \cdots$$

$$0 + 0 + 0 + \cdots = 1 + 0 + 0 + \cdots$$

$$0 = 1$$

Of course this does not prove that one equals zero. It proves that we must be very careful in dealing with infinite series. Next let us have another infinite series where it is perfectly clear that the terms may be regrouped into any order without fear of paradoxical results. Let a pie be divided into halves. Then let one of the halves be divided in two, giving quarters. Then of the two quarters can be divided into two eighths. Continue likewise. The infinite series is 1/2,1/4,1/8,1/16,$\cdots$. No matter how the pieces are rearranged, they should all fit back into the pie plate and exactly fill it.

The danger of infinite series is not they have an infinite number of terms but that they may sum to infinity. Safety is assured if the sum of the absolute values of the terms is finite. Such series are called ***absolutely convergent***.

Now consider an example from time-series analysis. The expression 1(1 - 2Z) can be expanded into powers of Z in (at least) two different ways. We have

$$\frac{1}{1-2Z} = 1 + 2Z + 4Z^2 + 8Z^3 + \cdots$$

$$= -\frac{1}{2Z}\,\frac{1}{1-\frac{1}{2Z}} = \frac{-1}{2Z}\left[1 + \frac{1}{2Z} + \frac{1}{4Z^2} + \cdots\right]$$

Which of the two infinite series is convergent depends on the numerical value of Z. Numerical values of Z which are of particular interest are $Z = +1$, $Z = -1$, and all those complex values of Z which are unit magnitude, say $|Z| = 1$ or $Z = \exp(i\omega\Delta t)$ where ω is the real Fourier transform variable. For such values the first series is divergent, but the second converges. So the only acceptable filter is ***anticausal***. Can we say that a series expansion is unique? To do so, we must demand that it converges. Complex-variable theory considers this with greater depth.

But books on complex-variable theory generally fail to point out the interpretation of infinite series as Z-transforms of time functions on the domain of discretized time. So students may fail to recognize an important connection between analyticity theory and causality theory which is this: Start from a series expansion containing both positive and negative powers of Z. To demand that this series converges ***on*** the unit circle $|Z| = 1$ is to demand only that the time function has finite energy. To further demand that the series converge everywhere ***inside*** the unit circle $|Z| \le 1$ (on the disk $|Z| \le 1$) forces the function to be causal since any inverse power of Z blows up at $Z = 0$.

Now go a little further and define an inverse $A = 1/B$. Whether A is causal is a question if $1/B$ converges in the disk. What about the opposite case where B happens to ***vanish*** somewhere in the disk? For example, $B = 1 - 2Z$ vanishes at $Z = 1/2$. There $A = 1/B$ must be infinite, that is to say, the series A must be non-convergent at $Z = 1/2$. So a_t would be non-causal. A most interesting case, called ***minimum phase*** is when both a filter and its inverse are causal. In summary

causal	$\|B(Z)\| < \infty$ for $\|Z\| \le 1$
causal inverse	$\|1/B(Z)\| < \infty$ for $\|Z\| \le 1$
minimum phase	both above conditions

Review of Impedance Filters

Use Z-transform notation to define a filter R(Z), its input X(Z), and output Y(Z). Then

$$Y(Z) = R(Z)\,X(Z)$$

The filter R(Z) is said to be ***causal*** if the series representation of R(Z) has no negative powers of Z. In other words, y_t is determined from present and past values of x_t. Additionally, the filter R(Z) will be ***minimum phase*** if 1/R(Z) has no negative powers of Z. This means that x_t can be determined from present and past values of y_t by straightforward polynomial division in

$$X(Z) = \frac{Y(Z)}{R(Z)}$$

Given that R(Z) is already minimum phase, it can additionally be an impedance function if positive energy or work is represented by

$$0 \le \text{work} = \sum_t \text{force} \times \text{velocity} = \sum_t \text{voltage} \times \text{current}$$

$$= \frac{1}{2} \sum_t (\bar{x}_t y_t + \bar{y}_t x_t)$$

$$= \text{coef of } Z^0 \text{ of } \left[\bar{X}\left(\frac{1}{Z}\right) Y(Z) + \bar{Y}\left(\frac{1}{Z}\right) X(Z)\right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{Re}\,(\bar{X}\,Y)\,d\omega$$

$$= \int \text{Re}\,(\bar{X}\,R\,X)\,d\omega = \int \bar{X}\,X\,\text{Re}\,(R)\,d\omega$$

Since $\bar{X}X$ could be an impulse function located at any ω, it therefore follows that $\text{Re}\,[R(\omega)] \ge 0$ for all real ω. In summary,

Conditions for a function to be an impedance function:	
causality	$r_t = 0$ for $t < 0$ OR $\lvert R(Z)\rvert < \infty$ for $\lvert Z\rvert \le 1$
causal inverse	$\lvert 1/R(Z)\rvert < \infty$ for $\lvert Z\rvert \le 1$
real part of F.T. is positive	$\mathrm{Re}\,[R(\omega)] = R(Z) + \bar{R}(1/Z) \ge 0$ real ω

Adding an impedance function to its Fourier conjugate we get a purely positive function (imaginary part is zero) like a power spectrum, say

$$\left(r_0 + r_1 Z + r_2 Z^2 + \cdots\right) + \left(\bar{r}_0 + \bar{r}_1 \frac{1}{Z} + \bar{r}_2 \frac{1}{Z^2} + \cdots\right) \ge 0 \quad \text{for real } \omega$$

$$R(Z) + \bar{R}\left(\frac{1}{Z}\right) \ge 0 \quad \text{for real } \omega$$

which is the basis for the statement that "the impedance time function is one side of an autocorrelation function."

Impedances also arise in economic theory where X and Y are price and sales volume. Then I suppose that the positivity of the impedance means that in the game of buying and selling you are bound to lose!

Causal Integration

Begin with a time function p_t. We define its Z-transform by

$$P(Z) = \cdots p_{-2} Z^{-2} + p_{-1} Z^{-1} + p_0 + p_1 Z + p_2 Z^2 + \cdots$$

Define an operator $-i\omega\Delta t$ by

$$\frac{1}{-i\hat{\omega}\Delta t} = \frac{1}{2}\,\frac{1 + Z}{1 - Z}$$

We define another time function q_t with Z-transform $Q(Z)$ by applying the operator to P

$$Q(Z) = \frac{1}{2}\frac{1+Z}{1-Z}P(Z)$$

Multiply both sides by (1-Z)

$$(1-Z)\,Q(Z) = \frac{1}{2}(1+Z)\,P(Z)$$

Equate the coefficient of Z^t on each side

$$q_t - q_{t-1} = \frac{p_t + p_{t-1}}{2}$$

Taking p_t to be an impulse function we see that q_t turns out to be a step function, that is,

$$p = \cdots 0,0,1,0,0,0,\cdots$$

$$q = \cdots 0,0,\tfrac{1}{2},1,1,1,\cdots$$

So q_t is the discrete domain representation of the integral of p_t from minus infinity to time t. It is the same as a Crank-Nicolson style integration of the differential equation dQ/dt = P. The operator (1+Z)/(1-Z) is called the bilinear transform. The accuracy of the approximation was investigated in another lecture entitled "Frequency Dispersion and Wave-Migration Accuracy." The conclusion was that $\hat{\omega}\Delta t/2 = \tan(\omega\Delta t/2)$.

We may note that this integration operator has a pole at Z = 1 which is exactly on the unit circle. This raises the possibility of the paradox of infinity. In other words there are other non-causal expansions too. To avoid any ambiguity we introduce a small positive number $\epsilon = 1 - \rho$. Now the integration operator becomes

$$I = \frac{1}{2}\frac{1+\rho Z}{1-\rho Z}$$

$$= \frac{1}{2}(1+\rho Z)\left[1 + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots\right]$$

$$= \frac{1}{2} + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots$$

Because ρ is slightly less than one this series converges for any value of Z on the unit circle. If we had chosen a small negative ϵ instead of a positive one we would have found it necessary to make an expansion in negative powers of Z instead of positive powers.

Now the big news is that the causal integration operator is an example of an impedance function. It is clearly causal with a causal inverse. Let us check in the frequency domain that the real part is positive. Rationalizing the denominator we have

$$I = \frac{1}{2}\frac{(1 + \rho Z)}{(1 - \rho Z)}\frac{(1 - \rho/Z)}{(1 - \rho/Z)} = \frac{(1 - \rho^2) + \rho(Z - 1/Z)}{\text{positive}}$$

$$= \frac{(1 - \rho^2) - 2i\rho \sin \omega\Delta t}{\text{positive}}$$

Again it is our choice of a positive ϵ which has caused $1 - \rho^2$, hence the real part to be positive for all ω.

As multiplication by $-i\omega$ in the frequency domain is associated with differentiation d/dt in the time domain, so is division by $-i\omega$ associated with integration. Now the surprising thing is that people usually associate the asymmetric operator (1,-1) with differentiation, but the inverse to the causal integration operator, namely

$$I^{-1} = 2\,\frac{1 - \rho Z}{1 + \rho Z}$$

$$= 2 - 4\rho Z + 4(\rho Z)^2 - 4(\rho Z)^3 + \cdots$$

is completely causal, not at all asymmetric, and also represents differentiation. That is to say, when the time sampling Δt tends to zero or, what is the same thing, when the frequency is sufficiently far from the folding frequency (where there is a pole), the operator I^{-1} represents differentiation. In fact, in linear systems analysis this is often the preferred discrete representation of differentiation. By analogy with the words *definite*

integral this operator may be called the ***definite derivative***. As we will see, the construction of higher-order stable differential equations must now be subject to the rules which we developed for combining impedance functions.

Occasionally it will be necessary to have a ***negative*** real part for the differentiation operator. This can be achieved by taking ϵ negative which means taking $\rho > 1$ and doing the infinite series expansion in powers of Z^{-1}, that is, anticausally instead of causally with positive powers of Z. In either case the imaginary part will be $-i\omega$ but the real part has opposite sign.

Functional Analysis

We will establish, in sequence, the following theorems about exponentials, logarithms and powers of Fourier transforms of filters:

1. The exponential of a causal filter is causal.
2. The exponential of a causal filter is minimum phase.
3. The logarithm of a minimum phase filter is causal.
4. Any real power of a minimum phase filter is minimum phase.
5. Any fractional power $-1 \leq \rho \leq 1$ of an impedance function is an impedance function.

To establish Theorem 1 we define the Z-transform of an arbitrary causal function

$$U(Z) \quad = \quad u_0 + u_1 Z + u_2 Z^2 + \cdots \qquad (1)$$

and substitute it into the familiar power series for exponential

$$B(Z) \quad = \quad e^U \quad = \quad 1 + U + \frac{U^2}{2} + \cdots \qquad (|U| < \infty) \qquad (2)$$

It is clear that no negative powers of Z will be generated so that B(Z) is also causal.

To establish Theorem 2, that the exponential is not just causal but also minimum phase, we consider

$$B_+ \quad = \quad e^{+U} \tag{3a}$$

$$B_- \quad = \quad e^{-U} \tag{3b}$$

Clearly both B_+ and B_- are causal and they are inverses of one another. A minimum phase filter is defined to be causal with a causal inverse. So B_+ and B_- are minimum phase.

Now we set out to establish the converse theorem, namely Theorem 3, that the logarithm of a minimum phase filter is causal. Equate the Z-derivative of (1) to the Z-derivative of the logarithm of (2):

$$\frac{dU}{dZ} \quad = \quad u_1 + 2u_2 Z + 3u_3 Z^2 + \cdots \tag{4a}$$

$$U \quad = \quad \ln B \tag{4b}$$

$$\frac{dU}{dZ} \quad = \quad \frac{1}{B}\frac{dB}{dZ} \tag{4c}$$

Since we assume B is minimum phase it means that both 1/B and dB/dZ on the right of (4c) are causal. Since the product of two causals is causal, we have dU/dZ causal. But clearly dU/dZ could not be causal unless U is causal. That proves it except for the remote danger that B might converge while dB/dZ diverges.

On to Theorem 4, which says that any real power of a minimum-phase function is minimum phase. Consider

$$B' \quad = \quad B^r \quad = \quad \left(e^{\ln B}\right)^r \quad = \quad e^{r \ln B} \tag{5}$$

Since B is assumed minimum phase, ln B by Theorem 3 will be causal. Scaling by a real constant r does not change causality. Exponentiating shows, by Theorem 2, that B^r is minimum phase.

Finally we will prove Theorem 5, that an impedance function can be raised to any fractional power $-1 \leq \rho \leq +1$ and the result is still an impedance function. An impedance function is defined as a minimum-phase function with the additional property that the real part of its Fourier transform is positive. This means that the phase angle ϕ lies in the range $-\pi/2 < \phi < +\pi/2$. Raising the impedance function to the ρ power will compress the range to $-\pi\rho/2 < \phi < \pi\rho/2$. This will keep its real part positive. Theorem 4 states that *any* power of a minimum-phase function is causal, which is a lot more than we need to be certain that a *fractional* power of an impedance function will be causal.

Rules for Compounding Impedance Functions

One of the difficulties in applied geophysics is this: Results may have physical utility only in a certain limited range of frequencies, and reasonable approximations may be made in that range. But if a spectrum or impedance becomes negative outside the applicable range, say near the Nyquist folding frequency, then the calculation (by Murphy's Law) will be unstable and hence useless. Thus Muir's rules[1] for compounding impedance functions deserve careful attention. Let R' denote a new impedance function generated from old known impedance functions R, R_1, or R_2. Muir's rules are:

i1:	Multiplication by positive scalar a	$R' = a R$
i2:	Inversion	$R' = \frac{1}{R}$
i3:	Addition	$R' = R_1 + R_2$

[1]Personal communication with Francis Muir.

Proofs:

i1: Obviously preserves causality and positivity of real parts of F.T.

i2: Causality is OK since by definition every impedance is minimum phase. Positivity follows since for any ω we have $1(a+ib) = (a-ib)/(a^2+b^2)$.

i3: Causality and positivity are trivial. Proof that the inverse will be causal is harder and will be done next.

The easiest proof is based on FGDP where the minimum-phase property is shown to mean that the function is causal and in the frequency domain if the phase curve does not loop around the origin. Positivity of the real part ensures that the phase does not loop about the origin. FGDP assumes polynomials rather than infinite series, but if wavelets really are transient this should cause no problem since truncation should then always be reasonable.

An abstract though excellent proof is found in complex variable theory. The analyticity inside the circle of R_1 and R_2 implies that the sum is analytic there. Positivity of the real part on the circle and LaPlace's equation inside implies that $Re(R_1 + R_2)$ does not vanish inside, so $|1/(R_1 + R_2)| < \infty$ inside $|Z| \leq 1$, which suffices.

Isomorphism with Reflectance Function C

Given any impedance function R, then the following equation defines an associated reflectance function C

$$C = \frac{1 - R}{1 + R} \tag{6a}$$

We will see that the reflectance function is also causal and that it is less than unity in magnitude, say

$$|C|^2 = \bar{C}\left(\frac{1}{Z}\right) C(Z) < 1$$

Causality follows because the numerator $1 - R$ is causal and the denominator, being the sum of two impedance functions, has a causal inverse. The product of two causals is causal. That the magnitude of C is less than unity follows from noting that the magnitude of the numerator is less than the magnitude of R and the magnitude of the denominator is greater. Unlike the impedance function $R(Z)$, the reflectance function $C(Z)$ is not necessarily minimum phase. An example is $R = 1 + Z/2$, $C = -.5Z/(2 + Z/2)$.

Equation (6a) may be solved for R:

$$R = \frac{1 - C}{1 + C} \tag{6b}$$

We may not inquire if C = causal and $|C| < 1$ alone will ensure that R is an impedance function.

Multiply (6b) on top and bottom by $1 + \bar{C}$:

$$R = \frac{(1 - C)(1 + \bar{C})}{|1 + C|^2}$$

$$= \frac{(1 - \bar{C}C) + (-C + \bar{C})}{\text{positive}}$$

$$= (\text{real}) + (\text{imag})$$

Clearly the positive reality is ensured by $|C| < 1$. The causality follows since the numerator of (6b) is assumed causal and the denominator is causal with positive real part (since $1 > |C|$). In summary, then, equation (6b) will reliably produce an impedance function from any apparent reflectivity function.

Example: ***Wide-Angle Wave Extrapolation***

Let $s = -i\hat{\omega}$ denote the causal positive discrete representation of the differentiation operator, say

$$s = -i\hat{\omega}\Delta t = 2\,\frac{1 - \rho z}{1 + \rho z}$$

Consider the following recursion starting from $S_0 = s$:

$$S_{n+1} = s + \frac{X^2}{s + S_n}$$

F. Muir introduced this recursion as a means of developing wide-angle square-root approximations for migration and developed his three rules i1,2,3 to show that every S_n is an impedance function. To see why this works, first note that the denominator $s + S_n$ is, for n = 0, the sum of two impedance functions. Then its inverse is an impedance function, and multiplication by the real positive constant X^2 and addition of another s all preserve the properties of impedance functions. As N becomes large this recursion either converges or it does not. Supposing that it does, we can see to what it converges by setting $S_{n+1} = S_n = S_\infty = S$. We have

$$S = s + \frac{X^2}{s + S}$$

$$S(s + S) = s(s + S) + X^2$$

$$S^2 = s^2 + X^2$$

$$S = (s^2 + X^2)^{\frac{1}{2}}$$

$$S = s\left(1 + \frac{X^2}{s^2}\right)^{\frac{1}{2}}$$

In wave extrapolation problems X^2 is $v^2k_x^2$ where v is the wave velocity and k_x is horizontal spatial frequency, namely, the Fourier dual to the horizontal x-axis. The quantities S_n are ik_z where k_z is the Fourier dual to the depth z-axis. The cases n = 0, 1, and 2 are commonly referred to as the 5-degree, 15-degree, and 45-degree equations, respectively. The desirability of S being positive real is related to the fact that it is acceptable for $\exp(ik_z z)$ to decay with z (when k_z is complex), but growth is almost certainly not acceptable.

Exact Square Root

The general form for stable extrapolation problems seems to be

$$\frac{dP}{dz} = -RP \tag{7}$$

where convergence is assured by the positive real part of the impedance function R. In reflection seismology there is great interest in the square-root extrapolation operator

$$R = -ik_z = \frac{-i\omega}{v}\left[1 - \frac{v^2 k_x^2}{\omega^2}\right]^{\frac{1}{2}} \tag{8}$$

At the moment we are disinterested in the space- or frequency-dependence of velocity, so we set $v = 1$, obtaining

$$R = [(-i\omega)^2 + k^2]^{\frac{1}{2}} \tag{9}$$

In (9) we would like a causal representation of the differentiation operator such as either of the following:

$$-i\hat{\omega} = \begin{cases} \dfrac{2}{\Delta t}\dfrac{1-\rho Z}{1+\rho Z} & -1 \ll \rho < 1 \quad \text{and} \quad Z = e^{i\omega\Delta t} \\ -i\omega + \epsilon & \epsilon > 0 \end{cases} \tag{10a,b}$$

We intend to establish that the following operator is an impedance function

$$R = [(-i\hat{\omega})^2 + k^2]^{\frac{1}{2}} \tag{11}$$

First note that $(-i\hat{\omega})$ is causal by (10), which means that $(-i\hat{\omega})^2$ is also causal. Also, k^2 is a delta function at the time origin. Thus R given by (11) is causal. Next, let us look at the phase. Figure 1 shows how the phase of (11) is constructed from its constituents.

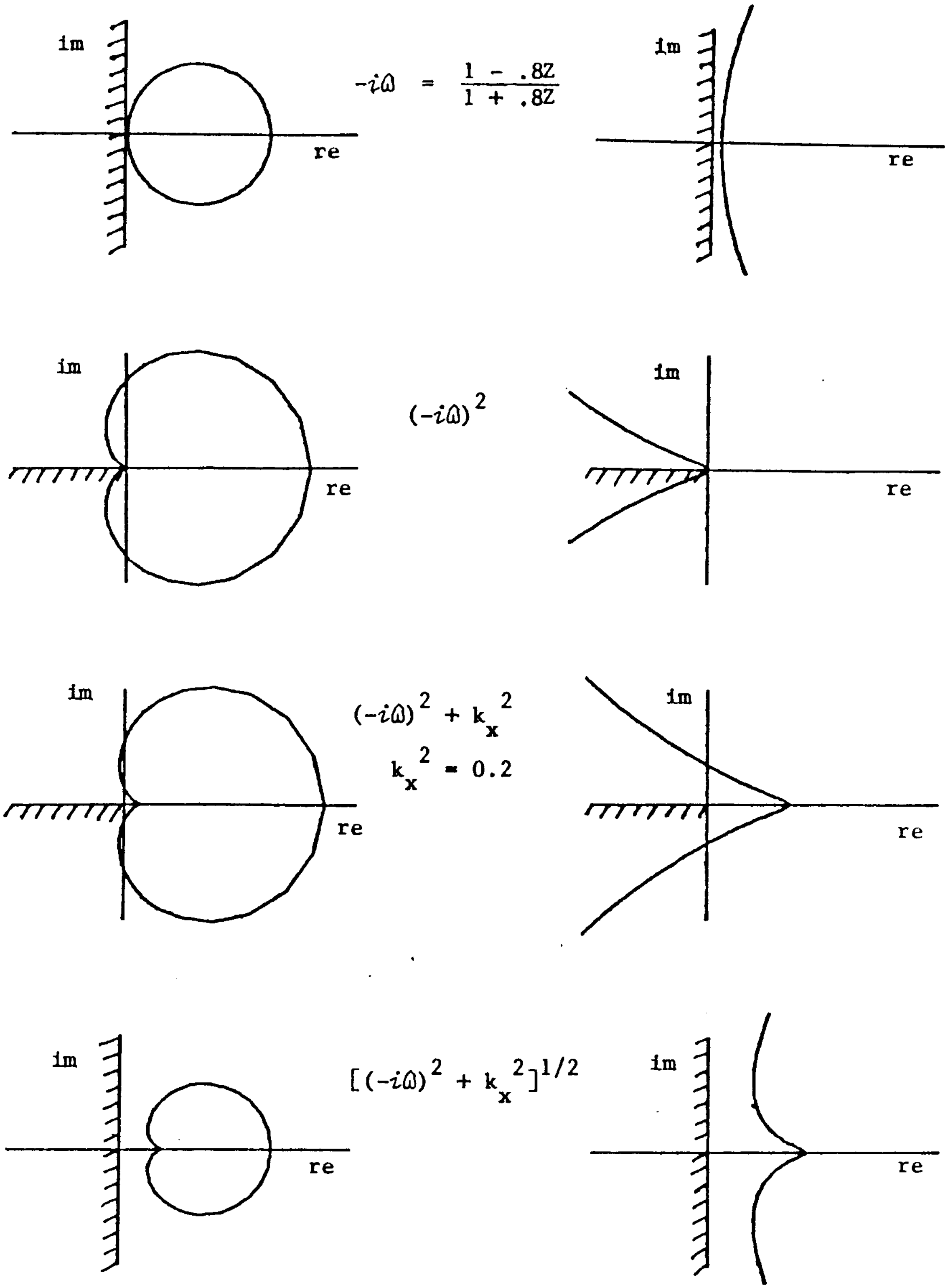

FIG. 1. Complex plane diagram of constituents of the extrapolation operator R as given by (11). The right column is the same as the left column blown up five times.

Now we have seen that R^2 is causal and that its phase has the "branch cut" property. That is, the phase of R has the positive real property. One of the aspects of minimum phase is that the phase does not loop around the origin. This is easily seen by inspecting

$$B = e^{U(Z)} = \exp\left[\sum_k^N U_k \cos k\,\omega + i \sum_k^N U_k \sin k\,\omega\right]$$

$$= \exp[r(\omega) + i\,\phi(\omega)]$$

Here the phase is a periodic function of ω, which means that in the plane of (Re B, Im B) the curve representing $B(\omega)$ does not enclose the origin. The branch cut forces R^2 to have this property and hence be minimum phase. Theorem 4 forces R to be causal and minimum phase. That, with the phase defined by figure 1, proves that R, given by (9), is an impedance function. (Muir previously established that some rational approximations to R are impedance functions, but the proof does not extend to the evanescent region of the square root.)

Fractional Integration and Constant Q

By equation (6) and Theorem 5 we know that fractional powers of integration and differentiation are also impedance functions. In fact, Kjartansson (1979) has advocated the fractional power as a stress-strain law for rocks. The conventional rock-mechanics studies begin with a stress-strain law such as

$$\text{stress} = \text{stiffness} \times \text{strain} + \text{viscosity} \times \text{strain-rate}$$

which in the transform domain is

$$\text{stress} = [(-i\omega)^0 \times \text{stiffness} + (i\omega)^1 \times \text{viscosity}]\ \text{strain} \qquad (12)$$

Without for the moment considering the physics of the matter, we can consider replacing the arithmetic average of the two terms by a geometric average, say

$$\text{stress} = \text{const} \times (i\omega)^{\epsilon} \text{ strain} \tag{13a}$$

$$= \text{const} \times (i\omega)^{\epsilon-1} \text{ strain-rate} \tag{13b}$$

where ϵ close to zero gives elastic behavior and ϵ close to one gives viscous behavior. The fact that $(-i\omega)^{\epsilon-1}$ is an impedance function meshes nicely with the concepts that (1) stress may be determined from strain history and strain may be determined from stress history, and (2) stress times strain-rate is dissipated power. Kjartansson (1979) points out that $(-i\omega)^{\gamma}$ exhibits the mathematical property called ***constant Q***, so that as a stress/strain law for fitting experimental data on rocks, it is far superior to the arithmetic average. To see the constant Q property more clearly, let us express $(-i\omega)^{\gamma}$, in real and imaginary parts:

$$(-i\omega)^{\gamma} = |\omega|^{\gamma} [e^{-i\pi \text{ sgn}(\omega)/2}]^{\gamma}$$

$$= |\omega|^{\gamma} \left\{\cos\left[\frac{\pi\gamma}{2} \text{ sgn}(\omega)\right] - i \sin\left[\frac{\pi\gamma}{2} \text{ sgn}(\omega)\right]\right\}$$

$$= |\omega|^{\gamma} \left[\cos\left(\frac{\pi\gamma}{2}\right) - i \text{ sgn}(\omega) \sin\left(\frac{\pi\gamma}{2}\right)\right] \tag{14}$$

The constant Q property follows from the constant ratio between the real and and imaginary parts of this function. Unfortunately, we have been unable to find a closed form representation for $(-i\omega)^{\gamma}$ in the discrete time domain. Kjartansson (1979) gives the form in the continuum as

$$\text{IFT}(-i\omega)^{\gamma} = \frac{\gamma}{\Gamma(1-\gamma)} t^{-1-\gamma} \qquad t > 0 \tag{15}$$

$$= 0 \qquad t < 0$$

Although γ is permitted to range from -1 to $+1$, singularities at $t = 0$ may need to be considered separately.

The log integration operator is one side of the Hilbert Transform.

Since the causal integral (6) is an impedance function, by Theorem 2 it should have a causal logarithm. Defining its logarithm as U we have

$$U(Z) \quad = \quad \ln \frac{1}{-i\hat{\omega}} \quad = \quad \ln \frac{\Delta t}{2} \frac{1 + \rho Z}{1 - \rho Z} \tag{16}$$

To obtain a time-domain representation of U we proceed as suggested by equation (4) and take the Z-derivative of any causal Z-transform with

$$\frac{dU}{dZ} \quad = \quad u_1 + 2u_2 Z + 3u_3 Z^2 + 4u_4 Z^3 + \cdots \tag{17}$$

Applying d/dZ to the right-hand side of (16) we get

$$\begin{aligned} \frac{dU}{dZ} &= \frac{d}{dZ}\left[\ln(\Delta t/2) + \ln(1 + \rho z) - \ln(1 - \rho z)\right] \\ &= \frac{\rho}{1 + \rho Z} + \frac{\rho}{1 - \rho Z} \\ &= 2\rho\left[1 + (\rho Z)^2 + (\rho Z)^4 + \cdots\right] \end{aligned} \tag{18}$$

Take the limit $\epsilon \to 0$ where $\rho = 1 - \epsilon$ and identify coefficients of like powers of Z in (17) and (18). Also substitute $Z = 0$ in (16) to find u_0. We have

$$u_k \quad = \quad \begin{cases} 0 \quad \text{for} \quad k \quad \text{negative} \\ \ln(\Delta t/2) \quad \text{for} \quad k = 0 \\ 2/k \quad \text{for} \quad k = 1,3,5,7,\ldots \\ 0 \quad \text{for} \quad k = 2,4,6,8,\ldots \end{cases} \tag{19}$$

What we see is that in the time domain the function $\ln[1/(-i\hat{\omega})]$ is causal and drops off as inverse time. This is just like one side of the Hilbert Transform including the discrete domain representation as inverse odd integers. In the frequency domain we have

$$\ln\left(\frac{1}{-i\omega}\right) = -\ln(-i\omega) = -\left[\ln|\omega| - i\,\frac{\pi}{2}\,\mathrm{sgn}(\omega)\right]$$

$$= -\ln|\omega| + i\,\frac{\pi}{2}\,\mathrm{sgn}(\omega) \qquad (20)$$

Adding (19) to the negative of its time reverse yields the Hilbert kernel $2/k$ for k odd. The corresponding operation on (20) naturally gives the imaginary sgn function.

$$\ln\left(\frac{1}{-i\omega}\right) - \ln\left(\frac{1}{i\omega}\right) = i\,\pi\,\mathrm{sgn}(\omega) \qquad (21)$$

The Hilbert kernal is an asymmetric time function with 90-degree phase shift and no color change. The log integral is causal with slight color change and phase shift about 90 degrees in the vicinity of $|\omega| = 1$. [Do not be confused by the differing scale factor of 2 between (20) and (21). When $|\omega| = 2$, both are imaginary and odd, so that both have the same 90° phase shift.]

Reflection from Q Contrast

Reflections arise at an interface of impedance with a well-known reflection strength

$$C = \frac{R_2 - R_1}{R_2 + R_1} \qquad (22)$$

We often think of the impedance as the velocity-density product, but at non-vertical incidence the product is divided by the angle cosine of the ray. We know that $(-i\omega)^{\epsilon}$ is also an impedance function, and we may suspect that it too could be inserted into (22) as, say, R_2 with, say, $R_1 = 1$. This gives

$$C = \frac{(-i\omega)^{\epsilon} - 1}{(-i\omega)^{\epsilon} + 1} \qquad (23)$$

Kjartansson (SEP-16, p. 131-140) has shown that this will describe the physics of a wave reflected in a medium of one constant Q value from another medium. Equation (23) is also the first term in an expansion for logarithm, and as ϵ tends to zero the expansion is dominated by the first term. Thus, the reflected wave takes the form

$$C = \frac{\epsilon}{2} \log\left(\frac{1}{-i\omega}\right) \tag{24}$$

which is expressed in the time domain by equation (19).

REFERENCES

Kjartansson, E., 1979, Constant Q - wave propagation and attenuation: J. Geophys. Res., v. 84, p. 4737-4748.

Exercises

1. Take $\epsilon < 0$ and expand the integration operator for negative powers of Z. Explain the sign difference.

2. The word "isomorphism" means not only that any impedance R_1, R_2, R' can be mapped into three rules for combining reflectances.

   a. What are these three rules?

   b. Although $C' = C_1 C_2$ does not turn out to be one of the three rules it is obviously true. Show either than it is a consequence of the three rules or conclude that it is an independent rule which can be mapped back into the domain of the impedances to make a fourth rule.

3. Consider the fourth-order Taylor expansion for square root in an extrapolation equation

$$\frac{dP}{dz} = i\omega\left[1 - \frac{1}{2}\left(\frac{vk}{\omega}\right)^2 - \frac{1}{8}\left(\frac{vk}{\omega}\right)^4\right] P$$

a. Will this equation be stable for the complex frequency $-i\omega = -i\omega_0 + \epsilon$? Why?

b. Consider causal and anticausal time-domain calculations with the equation. Which, if any, is stable?

4. Consider material velocity which may depend on frequency ω and on the horizontal x-coordinate as well. Suppose that luckily the velocity can be expressed in factored form $v(x,\omega) = v_1(x)\ v_2(\omega)$. Obtain a stable 45-degree wave-extrapolation equation. Hints: Try

$$s = -\frac{i\omega}{v_2}$$

$$X^2 = \text{positive eigenvalue of } (v_1\partial_x)(v_1\partial_x)^T$$

5. Is the Levinson Recursion in FGDP related to the rules in this paper? If so, how? Hint: See Wall's book on continued fractions.