SPLITTING AND FULL SEPARATION

The splitting method is a useful technique for obtaining numerical solutions to partial-differential equations. One motivation for this method is that for the 3-D, 15-degree, wave-extrapolation equation, we have no other stable approach with reasonable costs. Splitting is also the method of choice with 2-D wave extrapolation in laterally variable media. On a deeper level, splitting provides increased understanding of how to formulate inverse problems.

Splitting

The splitting method applied to numerical solution of the heat-flow equation

$$\frac{\partial T}{\partial t} = \sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \tag{1}$$

is to replace it by two different equations, each of which is used on alternating time steps:

$$\frac{\partial T}{\partial t} = 2\sigma \frac{\partial^2 T}{\partial x^2} \qquad \text{(all } y\text{)} \tag{2a}$$

$$\frac{\partial T}{\partial t} = 2\sigma \frac{\partial^2 T}{\partial y^2} \qquad \text{(all } x\text{)} \tag{2b}$$

The occurrence of the numerical factor of 2 in (2a) and (2b) is something of a mathematical convention because, as it stands, the equations do not make it clear that σ vanishes over half the time steps, say, σ vanishes over even-numbered time steps in (2a) and odd-numbered time steps in (2b). It can be proved mathematically that the solution to (2a,b) converges to the solution to (1) with errors of the order of Δt. Hence the error goes to zero as Δt goes to zero. The motivation for the splitting method was developed in earlier

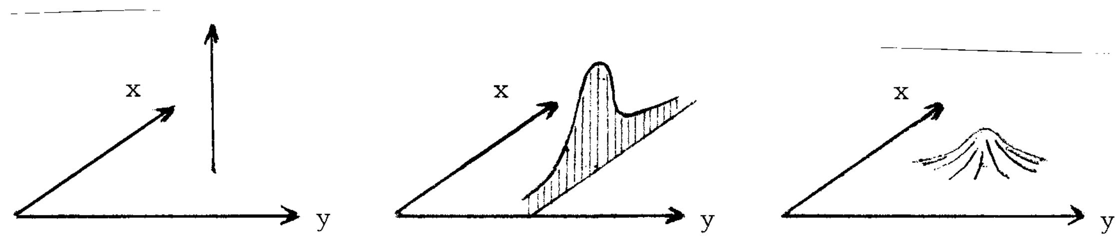

FIG. 1. Temperature distribution in the (x,z)-plane beginning from a delta function (left). After heat is allowed to flow in the x-direction but not in the y-direction we have the heat located in a sheet (center). Finally allowing heat to flow for the same amount of time in the y-direction but not the x-direction we get the same symmetrical Gaussian result that would have resulted had heat moved in x- and y-directions simultaneously (right).

mount up simply by counting multiplications. When the data base does not fit entirely into the random access memory, as is almost the definition of a ***large problem***, then each step of the splitting method demands that the data base be transposed, say from (x,y) storage order to (y,x) storage order. Transposing requires no multiplications, but in many environments it would be by far the most costly part of the whole computation. So if transposing cannot be avoided, at least it should be reduced to a practical minimum. Thus we can easily envision circumstances which dictate a middle road between splitting and separation. This would happen if σ were a slowly variable function of x or y. Then it might be found that although $\sigma\partial_{xx}$ does not strictly commute with $\sigma\partial_{yy}$, it comes close enough that a number of time steps may be made with (2a) before transposing the data and switching over to (2b). Questions like this but with more geophysical interest arise with the wave-extrapolation equation considered next.

Application to Lateral Velocity Variation

A circumstance where the degree of non-commutivity of two differential operators has a simple physical meaning and an obviously significant geophysical application is the so-called monochromatic 15-degree wave-extrapolation equation in inhomogeneous media. Taking $v \approx \bar{v}$ it is

$$\frac{\partial U}{\partial z} = \left\{ \frac{i\omega}{\bar{v}(z)} + i\omega\left[\frac{1}{v(x,z)} - \frac{1}{\bar{v}(z)}\right] - \frac{\bar{v}(z)}{2i\omega}\frac{\partial^2}{\partial x^2} \right\} U \tag{3}$$

$$= (\text{retardation} + \text{thin lens} + \text{diffraction})\ U$$

Inspection of (3) shows that the retardation term commutes with the thin-lens term and with the free-space diffraction term. But the thin-lens term and the diffraction term do not commute with one another. In practice it seems best to split, doing the thin-lens part analytically and the diffraction part by the Crank-Nicolson method. Then stability is assured because the stability of each separate problem is known. Also the accuracy of the analytic solution is an attractive feature. Now the question is, to what degree do these two terms commute?

The problem is just that of focusing a slide projector. Adjusting the focus knob amounts to repositioning the thin-lens term in comparison to the free-space diffraction term. There is a small range of knob positions over which no one can notice any difference, and a larger range over which the people in the back row are not disturbed by misfocus. Much geophysical data processing amounts to downward extrapolation of data. The lateral variation of velocity occurring in the lens term is known only to a limited accuracy. In fact we may be trying to determine $v(x)$ by the extrapolation procedure. For long lateral spatial wavelengths we may assume that the terms commute and that the effect of the poorly known lateral variation in v can be considered without regard to the diffraction. At shorter wavelengths the diffraction and lensing effects must be interspersed. So the real issue is not merely computational speed but the interplay between data accuracy and the possible range for velocity in the underlying model.

Application to 3-D Migration

The operator for migration of zero-offset reflection seismic data in three dimensions is expandable to second order by Taylor-series expansion to the so-called 15-degree approximation

$$\left[\frac{\omega^2}{v^2} + \partial_{xx} + \partial_{yy}\right]^{\frac{1}{2}} \approx \frac{-i\omega}{v} + \frac{v\partial_{xx}}{-2i\omega} + \frac{v\partial_{yy}}{-2i\omega} \tag{4}$$

Considering the most common case where v is slowly variable or independent of x and y, we see that the conditions of full separation do apply. This is good news because it means that we can use ordinary 2-D section migration programs for 3-D, migrating the in-line data and the out-of-line data in either order. The bad news comes when you try for more accuracy. Keeping more terms in the Taylor-series expansion soon brings in the cross term ∂_{xxyy}. Such a term allows neither full separation nor splitting, and we are left with explicit methods. Things are even worse with the rational fraction expansion where an attempt to get 45-degree accuracy leads to terms like (const $+ \partial_{xx} + \partial_{yy})\partial_z$ which cannot even be handled by the explicit method. Fortunately, present-day marine data acquisition techniques are sufficiently crude in the out-of-line direction that there is little justification for out-of-line processing beyond the 15-degree equation. But there may be justification with land data. Fourier transformation of at least one of the two space axes will solve the computational problem. This should be a good approach when the medium velocity does not vary laterally so rapidly as to invalidate application of Fourier transformation.

Non-Separability in Midpoint-Offset Space

Reflection seismic data gathering is done on the earth's surface. One can imagine the appearance of the data which would result if the data were generated and recorded at depth, that is, with deeply buried shots and geophones. Such buried data could be synthesized from surface data by first downward extrapolating the geophones, then using the reciprocal principle to interchange sources and receivers, and finally downward extrapolating the surface shots (now the receivers). A second, equivalent approach would be to march downward in steps, alternating between shots and geophones. This latter approach is investigated with more thoroughness in later chapters, but we may simply state the result as the equation

$$\frac{\partial U}{\partial z} = \left\{ \left[\frac{\omega^2}{v^2(s)} + \frac{\partial^2}{\partial s^2} \right]^{\frac{1}{2}} + \left[\frac{\omega^2}{v^2(g)} + \frac{\partial^2}{\partial g^2} \right]^{\frac{1}{2}} \right\} U \tag{5}$$

The equivalence of the two approaches has mathematical expression in the fact that the shot coordinate s and geophone coordinate g are independent variables, so the two square-root operators commute. The same solution is obtained by splitting as by full separation.

In a later chapter equation (5) is derived and then converted to the space of midpoint y and half-offset h, where it takes the form

$$\frac{\partial P}{\partial z} = \left\{ \left[\frac{\omega^2}{v^2} + \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial h} \right)^2 \right]^{1/2} + \left[\frac{\omega^2}{v^2} + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial h} \right)^2 \right]^{1/2} \right\} P \qquad (6)$$

It is not our purpose here to abstract the results of much future material but merely to note that the operator in (6) is *not* the sum of a midpoint operator and an offset operator. So migration and stacking are not interchangeable. It turns out that the accuracy of "conventional processing" can be analyzed by analyzing the y-h cross terms in (6). And "conventional processing" can be improved by organizing things to minimize the effect of the non-commutivity of the two operators.

Proof of the Validity of the Splitting and Full-Separation Concepts

Partial differential operators may be approximated by difference operators, which in turn may be represented by matrices. Then the property of commutivity ($AB = BA$), or its lack ($AB \neq BA$), gets carried from the differential operators to the matrices. Commutivity is very nice because it means that the operation of one stage of the solution process, say multiplication by $(I + \Delta zA)$, can be interchanged with the succeeding stage of the process, say $(I + \Delta zB)$. Thus regrouping may be done for computational convenience or for any conceptual advantage such as when formulating an inverse problem. Use of the Crank-Nicolson finite-differencing method changes only the fact that the matrix operator of any stage has the slightly different formal expression $(I - \Delta zA/2)^{-1}(I + \Delta zA/2)$.

To illustrate this further take A to be the operator $v\partial^2/\partial x^2$ and B to be the operator $v\partial^2/\partial y^2$. Reference to the 4-by-4, (x,y)-space of 16-point difference operators in the previous section shows that these operators have non-zero elements indicated by v where

$$\frac{\partial^2}{\partial x^2} \simeq \left[\begin{array}{cccc|cccc|cccc|cccc}
v&v&&&&&&&&&&&&&&\\
v&v&v&&&&&&&&&&&&&\\
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&&&&&&&&&&&&v&v&v&\\
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\end{array}\right] \tag{7a}$$

$$\frac{\partial^2}{\partial y^2} \simeq \left[\begin{array}{cccc|cccc|cccc|cccc}
v&&&&v&&&&&&&&&&&\\
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&&v&&&&v&&&&&&&&&\\
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\end{array}\right] \tag{7b}$$

Study (7a) to discover that it is tridiagonal with the additional property that some special elements on the superdiagonal and some on the subdiagonal are identically zero. Also, (7a) can be partitioned into 4-by-4 submatrices, and the existence of the special zeros enables us to say that only the diagonal blocks are non-vanishing. So in a block-matrix sense, (7a) is a diagonal matrix. The converse happens with (7b): its submatrices are diagonal but its blocks take a tridiagonal pattern. It is now apparent that if v is constant, the matrices representing ∂_{xx} and ∂_{yy} commute, as they should. Likewise, it is also apparent that incorporating a variable velocity $v(x,y)$ causes the matrices to be non-commutative just as the differential operators are non-commutative.

The validity of the splitting method is not based on the discretization of the x- and y-axes as we have done in the previous paragraph. It depends on the discretization of the z-axis and the fact that to first order in Δz we have

$$(1 + \Delta z v \partial_{xx})\ (1 + \Delta z v \partial_{yy}) \ = \ 1 + \Delta z v(\partial_{xx} + \partial_{yy}) \tag{8}$$

The error for a single step given by (8) is order Δz^2. The number of steps Nz required to get from z_1 to z_2 is just $Nz = (z_2 - z_1)/\Delta z$. Therefore, the accumulated error is proportional to Δz. Hence it vanishes in the limit, justifying the splitting method. The matrices (7a,b) do, however, illustrate that the complete question of commutivity involves boundary conditions as well as differential operators.

Brown's Article

More details may be found in SEP-15, p. 214-232, in an article entitled "Splitting and Separation of Differential Equations with Applications to Three-Dimensional Migration and Lateral Velocity Variation" by David Brown. Brown's paper contains Muir's representation of the 3-D migration operator, which is exact in the in-line and out-of-line direction but has only quadratic accuracy in between. His paper also discusses a pitfall, that a stable equation can be split into two unstable parts, and illustrates it by the 45-degree extrapolation equation in laterally variable media. (I nearly pushed Bert Jacobs into the pit!) I consider Brown's paper to be a landmark in geophysical data processing. Before Brown did his work, I had difficulty defining the nature of the problem with midpoint-offset coordinates. It is easy to say in retrospect that we all knew about the commutivity of differential operators and their exponentials, but the fact remains that solving the heat-flow equation by full separation was not mentioned in either Richtmyer's book on finite differences or Morse and Feshbach's book on Mathematical Physics. They too did it the hard way!