A POOR MAN'S GUIDE TO POINT SOURCES IN TWO DIMENSIONS

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Abstract

The problem of implementing 2-D algorithms in 3-D media where the geology varies in only two dimensions, is basically one of converting line sources and receivers to point sources and receivers. Exact solutions are ponderous, and computationally expensive. The stationary phase approximation, on the other hand, provides relatively simple modifications to the algorithms, and is accurate in the far-field, the domain of most seismic data.

Introduction

The perceptive and dedicated researcher has no doubt noticed that many of the formulas in this volume assume a two-dimensional world. The extension to full 3-D is trivial. For example, the Born approximation relating seismic reflection data D to a density-modulus potential becomes (using units where 2π's magically disappear)

$$\langle k_x, k_y, 0 | D | k_x', k_y', 0 \rangle = - \frac{\omega^2 \rho_0}{4v^2 \nu\nu'} \langle k_x, k_y, -\nu | V | k_x', k_y', \nu' \rangle \qquad (1)$$

where ν and ν', given by

$$\nu = \left(\frac{\omega^2}{v^2} - k_x^2 - k_y^2 \right), \qquad \text{and} \qquad \nu' = \left(\frac{\omega^2}{v^2} - k_x'^2 - k_y'^2 \right)$$

are the spatial frequencies in the z-direction, and the potential has elements

$$\langle k_x, k_y, -\nu | V | k_x', k_y', \nu' \rangle = a(k_x - k_x', k_y - k_y', -\nu - \nu')$$

$$- \frac{(k_x k_x' + k_y k_y' - \nu\nu')}{\omega^2} b(k_x - k_x', k_y - k_y', -\nu - \nu') \qquad (2)$$

The problem is that neither the 2-D nor the 3-D expressions is directly applicable to the seismic experiment, since no one seems to have the energy to do a full coverage for both source and receivers. The most typical thing to do is to place source and receivers along a line (say y= y'= 0), and then convert an underdetermined problem into an overdetermined one by assuming that velocity and density vary only in x and z.

There are two ways to deal with this seismic problem: the correct way, and the way we propose to do it. To do things the right way, you need merely to blow the dust off your book on integral transforms, and get to work, following the usual derivation of the Born approximation except to clamp y and y' at zero, instead of ignoring them or Fourier transforming them. The way we do it is a bit simpler. Starting with the 3-D form of the Born approximation, we assume that the modulus and density do not depend on y, then do a stationary phase approximation to get a Fourier-transform expression for y = y' = 0.

The Stationary Phase Approach

If modulus and density do not depend on y, then

$$a(k_x - k_x', k_y - k_y', -\nu - \nu') \to a(k_x - k_x', -\nu - \nu')\, \delta(k_y - k_y')$$

$$b(k_x - k_x', k_y - k_y', -\nu - \nu') \to b(k_x - k_x', -\nu - \nu')\, \delta(k_y - k_y')$$

so the quadruple Fourier transform of the data has the form

$$\langle k_x, k_y, o | D | k_x', k_y', 0 \rangle = - \frac{\omega^2 \rho_o}{4 v^2 \nu\nu'}\, \delta(k_y - k_y')$$

$$\left\{a(k_x-k_x',-\nu-\nu') - \frac{k_x k_x' - \nu\nu' + k_y^2}{\omega^2} b(k_x-k_x',-\nu-\nu')\right\}$$

We can inverse transform to get D back in the y, y' representation and then set y = y' = 0:

$$\langle k_x,0,0|D|k_x',0,0\rangle = \int dk_y \int dk_y' \langle k_x,k_y,0|D|k_x',k_y',0\rangle$$

$$= -\frac{\omega^2 \rho_o}{4v^2} \int dk_y \frac{1}{\nu\nu'}$$

$$* \left\{a(k_x-k_x',-\nu-\nu') - \frac{k_x k_x' - \nu\nu' + k_y^2}{\omega^2} b(k_x-k_x',-\nu-\nu')\right\} \qquad (3)$$

This equation doesn't look very suitable for inversion, there being an annoying integral standing between us and what we want. Here's where the stationary phase comes in. We express a and b as Fourier transforms over z.

$$\begin{pmatrix} a \\ b \end{pmatrix}(k_x-k_x',-\nu-\nu') = \int dz e^{i(\nu+\nu')z} \begin{pmatrix} a \\ b \end{pmatrix}(k_x-k_x',z)$$

We then invert the order of integration to get [in place of equation (3)]

$$\langle k_x,0,0|D|k_x',0,0\rangle = -\frac{\omega^2 \rho_o}{4v^2} \int dz \left\{a(k_x-k_x',z) \int dk_y \frac{e^{i(\nu+\nu')z}}{\nu\nu'}\right.$$

$$\left. - b(k_x-k_x',z) \int dk_y \frac{e^{i(\nu+\nu')z}}{\nu\nu'} \frac{k_x k_x' - \nu\nu' + k_y^2}{\omega^2}\right\} \qquad (4)$$

We perform the integration over k_y by assuming (1) that everything except the exponential is much more slowly varying than the exponent itself; and (2) that the main contributions to the integral come from the point where the phase of the exponent is stationary. These assumptions basically allow us to evaluate the integral by expansion about the point where the derivative of $\nu+\nu'$ with respect to k_y is zero, which in this case is the point $k_y = 0$. Thus, we can write

$$\nu + \nu' = k_z + k_y^2 k_z''$$

where

$$k_z = \left(\frac{\omega^2}{v^2} - k_x^2\right)^{\frac{1}{2}} + \left(\frac{\omega^2}{v^2} - k_x'^2\right)^{\frac{1}{2}} \equiv \nu_0 + \nu_0'$$

$$k_z'' = - \frac{k_z}{\left(\frac{\omega^2}{v^2} - k_x^2\right)^{\frac{1}{2}} \left(\frac{\omega^2}{v^2} - k_x'^2\right)^{\frac{1}{2}}} = - \frac{k_z}{\nu_0 \nu_0'}$$

Under these circumstances the integrals over k_y are easily done, giving

$$\langle k_x,0,0|D|k_x',0,0\rangle = - \frac{\omega^2 \rho_0}{4v^2} \left(ik_z \nu_0 \nu_0'\right)^{-\frac{1}{2}}$$

$$\left\{\tilde{a}(k_x-k_x',-k_z) - \frac{k_x k_x' - \nu_0 \nu_0'}{\omega^2} \tilde{b}(k_x-k_x',-k_z)\right\}$$

$$\equiv \tilde{C}_1(\tilde{a} + C_2 \tilde{b}) \tag{5}$$

where $\tilde{a}$ and $\tilde{b}$ are slightly gained versions of a and b:

$$\tilde{a}(x,z) = \frac{a(x,z)}{z^{\frac{1}{2}}} = \frac{1}{z^{\frac{1}{2}}}\left[\frac{K_0}{K(x,z)} - 1\right] \tag{6a}$$

$$\tilde{b}(x,z) = \frac{b(x,z)}{z^{\frac{1}{2}}} = \frac{1}{z^{\frac{1}{2}}}\left[\frac{\rho_0}{\rho(x,z)} - 1\right] \tag{6b}$$

We seem to have a slightly modified 2-D algorithm on our hands. The original 2-D algorithm was of the form [see An Inversion Method For Acoustic Wave Fields, equation (23)]

$$\langle k_x,0|D|k_x',0\rangle = C_1(a + C_2 b).$$

The transition from line sources to point sources has been accomplished by (a) changing the factor C_1 to

$$\tilde{C}_1 = \left(\frac{\nu_o \nu_o'}{ik_z}\right)^{\frac{1}{2}} C_1,$$

and (b) re-interpreting the potential functions a and b according to equation (6).

Of course, this result is only approximate. Seismic data being basically far-field, however, we expect it to be a pretty good one. As long as one is content to use the simple Born approximation it would be very difficult to justify the additional computational cost of a more accurate expression, though we suppose that if a full inversion of seismic data is to be attempted, the stationary phase approximation should not be used.