# SEISMIC INVERSION IN A LAYERED MEDIUM - THE GELFAND-LEVITAN ALGORITHM

*Robert H. Stolt and Bert Jacobs*

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There are two basic approaches to the seismic inverse problem. One is to transform the problem into one which has already been solved, which means expressing the unknown earth parameters in the form of a Schroedinger potential. The other is to develop an inversion theory fitted to the seismic problem as it stands.

Though the first option appears open only for a layered medium, it has been applied there with a vengeance. In 1969, Jerry Ware and Keiiti Aki published a formal inversion technique using the Gelfand-Levitan algorithm of quantum scattering theory. Though in the paper the technique was restricted to plane waves at normal incidence, Ware, in his Ph.D. thesis, extended it to plane waves at all sub-critical angles. Ware never published his extension, apparently because he was unable to invert data with post-critical reflections.

In this article we review the basic Gelfand-Levitan technique as developed by Ware for seismic data, and extend it to post-critical reflections. Hopefully, the discussion will clarify why the technique is applicable only to the layered medium, and why, incidentally, that may be just as well.

## *A. The Problem*

Suppose we have a layered medium with variable density $\rho(z)$ and velocity $v(z)$. For $z < 0$ $\rho$ and $v$ are constants $\rho_0$ and $v_0$. For large $z$, $\rho$ and $v$ asymptotically approach different constants $\rho_f$ and $v_f$. Since for any practical computational scheme we will run out of patience before running out of $z$, we will just say that $\rho = \rho_f$ and $v = v_f$ for $z > z_f$.

Consider the following experiment. A plane wave $\psi_i(\omega,p,z)$ of lateral wavenumber $p$ and frequency $\omega$ is incident on the variable velocity-density region from above (the region $z < 0$). The angle of incidence $\theta$ of this wave is $\mathrm{Asin}(pv_0/\omega)$. The experiment produces a reflected wave $\psi_r(\omega,p,z)$ in the region $z < 0$ and a transmitted wave $\psi_t(\omega,p,z)$ in the region $z > z_f$. The transmitted wave is never seen again, but the reflected wave can be seen and measured.

The problem is, given the results of many such experiments using waves of different frequency and lateral wavenumber, can the density and velocity functions $\rho(z)$ and $v(z)$ be uniquely determined?

## *B. A Schroedinger Form for the Wave Equation*

In a layered medium, the scalar wave equation for pressure may be written as an ordinary differential equation

$$\left[\frac{\partial}{\partial z}\frac{1}{\rho(z)}\frac{\partial}{\partial z} - \frac{p^2}{\rho(z)} + \frac{\omega^2}{K(z)}\right]\psi(\omega,p,z) = 0 \qquad \text{(B-1)}$$

where $\rho$ is density, $K$ is bulk modulus, $z$ is depth, $\omega$ is frequency, and $p$ is lateral spatial frequency or wavenumber. A one-dimensional Schroedinger equation has a somewhat different form:

$$\left(\frac{\partial^2}{\partial\tau^2} + \omega^2 - V(\tau)\right) \phi(\omega,\tau) = 0 \tag{B-2}$$

where the potential function $V(\tau)$ is independent of frequency and is a local function of $\tau$ (i.e. is not a differential operator). The Gelfand-Levitan technique (as we shall see) solves for potentials of the Schroedinger form given certain information at all $\omega$. Consequently, we wish to transform (B-1) so it looks like (B-2).

To do this, we need somehow to rescue $\rho^{-1}$ from its z-derivative sandwich, and at the same time remove K from underneath $\omega^2$. It turns out that there is more than one way to do this. We shall look at several, all of which involve a change of both dependent and independent variables.

We look at the simplest way first. Make a coordinate transformation $z \to \tau$ defined by $dz = v\, d\tau$ where $v$ is some function of $\tau$ (try not to think of $v$ as velocity just yet). Also make a transformation $\psi \to \phi = \eta\, \psi$, where $\eta$ is some other function of $\tau$. Putting these two transformations into (1) and multiplying by $K\,\eta$, we get

$$\left(\frac{K\eta}{v}\frac{\partial}{\partial\tau}\frac{1}{\rho v}\frac{\partial}{\partial\tau}\frac{1}{\eta} - \frac{Kp^2}{\rho} + \omega^2\right)\phi = 0 \tag{B-3}$$

If one remembers the identity

$$\frac{\partial}{\partial\tau}\eta^2\frac{\partial}{\partial\tau}\frac{\phi}{\eta} = \eta\frac{\partial^2}{\partial\tau^2}\phi - \phi\frac{\partial^2}{\partial\tau^2}\eta \,.$$

it is clear that the desired function for $\eta$ is

$$\eta = (\rho v)^{-\frac{1}{2}} \tag{B-4a}$$

Further, the coefficient of $(\partial^2)/(\partial\tau^2)\,\phi$ in (B-3) will equal 1, if

$$v = (K/\rho)^{+\frac{1}{2}} \tag{B-4b}$$

which, by golly, is the velocity. With the definitions (B-4a) and (B-4b), equation (B-3) takes the Schroedinger form (B-2), with

$$V(\tau) \rightarrow V(\tau,p) = p^2 v^2(\tau) + \frac{1}{\eta(\tau)} \frac{\partial^2}{\partial \tau^2} \eta(\tau) \qquad \text{(B-4c)}$$

Actually V also depends on the lateral wavenumber p, which really doesn't matter since p may be considered a fixed parameter. [That is, we solve for $V(\tau,p)$ for some fixed p, given data at all $\omega$.]

What does matter, however, is that for $p \neq 0$ the potential (B-4c) does not go to zero in a region of constant velocity and density. That in itself is not too big a concern; what is annoying is that regions of different constant velocity will produce different constant potentials. This is annoying because the Schroedinger equation (2) has propagating solutions only where $\omega^2 > V$. If $\omega^2 < V$ over a large region, then the solution $\phi$ must decay exponentially in that region. If $V_0 < V_f$ (as will normally be the case) there will exist experiments with an incoming and a reflected wave, but no transmitted wave. That is not a complete tragedy. As long as $v_0$ is the low velocity it will be seen that the Gelfand-Levitan method works just fine. If $v_0$ is not the lowest velocity, however, complications do arise.

This complication can be averted by choosing a slightly different transformation. Instead of considering p the fixed parameter in equation (B-1), define a new parameter

$$\alpha \equiv \frac{v_0 p}{\omega} \equiv \sin \theta_{IN}$$

$\alpha$ is just the sine of the angle made by the incident plane wave. Equation (B-1) then has the form

$$\left( \partial_z \frac{1}{\rho} \partial_z + \frac{\omega^2}{K_e} \right) \psi = 0 \qquad \text{(B-5)}$$

where $K_e$, an "effective" bulk modulus, is

$$K_e = \frac{K}{1 - \frac{\alpha^2 v^2}{v_0^2}} \qquad \text{(B-6a)}$$

It is pretty clear that if $\alpha v/v_0$ ever exceeds 1, something bad will happen. The experimenter, however, has control over $\alpha$, so provided he uses plane waves of a reasonable incidence angle, nothing bad will ever happen.

Equation (B-5) looks just like equation (B-1) without the $p^2$ term. If we now make the same transformations as was done to (B-1), ($dz \rightarrow d\tau = dz/v_e$, $\psi \rightarrow \phi = \psi/\eta_e$) we find that with

$$v_e = \sqrt{\frac{K_e}{\rho}} = \frac{v}{\sqrt{1 - \frac{\alpha^2 v^2}{v_0^2}}} \qquad \text{(B-6b)}$$

$$\eta_e = \frac{1}{\sqrt{\rho v_e}} \qquad \text{(B-6c)}$$

The wave equation takes the Schroedinger form (B-2) with

$$V(\tau) \rightarrow V(\tau,\alpha) = \frac{1}{\eta_e} \partial_\tau^2 \eta_e \qquad \text{(B-6d)}$$

This second transformation, which is the one suggested by Ware, appears to have traded one problem for another. If the potential $V(\tau,p)$ of equation (B-4c) could be solved for at two or more p-values, it would be a very simple matter to solve for $v(\tau)$, and reasonably simple to solve for $\rho(\tau)$. The potential of equation (B-6d) will be somewhat harder to unravel. Assuming we can solve for $V(\tau,\alpha)$ at two or more $\alpha$'s, and then solve equation (B-6d) in each case for $\eta_e(\tau)$, we still have a problem. Since $\tau = \tau(z,\alpha)$ is a different function of z for each $\alpha$, some processes analagous to velocity analysis and moveout correction must be applied before comparing the different $\eta_e$ and solving for $\rho$ and v. The second transformation seems to have separated the high- and low-frequency components of v much like normal processing.

Other Schroedinger-type equations can be formed from (B-1) in which p takes the place of $\omega$. In the simplest we take $\omega$ to be the fixed parameter, defining

$$\eta = \rho^{-\frac{1}{2}}$$

and

$$\phi = \eta \psi$$

Then we can write a Schroedinger equation for $\phi$:

$$[\partial_z^2 + E - V(z,\omega)] \phi(\omega,E,z) = 0 \qquad \text{(B-7)}$$

where

$$E = \frac{\omega^2}{v_{min}^2} - p^2$$

is a Schroedinger energy, and

$$V(\omega,z) = \omega^2 \left( \frac{1}{v_{min}^2} - \frac{1}{v^2} \right) + \frac{1}{\eta} \partial_z^2 \eta$$

is a Schroedinger potential. For a fixed $\omega$, a suite of experiments with $0 < |p| < \omega^2/v_{min}^2$ will give solutions to (B-7) for $E < \omega^2/v_{min}^2$. Provided $\omega$ is chosen large enough, the potential should be resolvable.

Alternately, the fixed parameter might be $\alpha$ again. Then (B-1) becomes

$$\left( \partial_z \frac{1}{\rho} \partial_z + \frac{p^2}{\gamma} \right) \psi = 0 \qquad \text{(B-8)}$$

with

$$\gamma = \frac{\rho}{\frac{v_o^2}{\alpha^2 v^2} - 1}$$

The same sort of transformation as before gives

$$[d_\zeta^2 + p^2 - V(\zeta,\alpha)]\ \phi(p,\alpha,\zeta) = 0 \qquad \text{(B-9)}$$

with

$$d\zeta = \frac{dz}{\lambda} \qquad \text{(B-10a)}$$

$$\lambda = \sqrt{\frac{\gamma}{\rho}} = \frac{1}{\sqrt{\frac{v_o^2}{\alpha^2 v^2} - 1}} \qquad \text{(B-10b)}$$

$$\phi = \eta\ \psi \qquad \text{(B-10c)}$$

$$\eta = (\lambda\rho)^{-\frac{1}{2}} \qquad \text{(B-10d)}$$

$$V(\zeta,\alpha) = \frac{1}{\eta}\ \eta_{\zeta\zeta} \qquad \text{(B-10e)}$$

This set-up has the joys and pitfalls of the Ware transformation, except that p in a real live seismic experiment is not likely to be as well-sampled as $\omega$.

In summary, there would seem to be at least four ways to put the scalar wave equation for a layered medium in Schroedinger form, namely frequency-wavenumber (f-k), frequency-dip (f-$\alpha$), wavenumber-frequency (k-f), and wavenumber-dip (k-$\alpha$). If post-critical reflections are to be excluded from the analysis, then (f-$\alpha$) would likely be the simplest choice. If they are to be included, (f-k) would seem to be best.

## C. *The Gelfand-Levitan Solution*

We have just seen a number of ways to transform the wave equation for a layered medium into a Schroedinger equation. We will now write down the Gelfand-Levitan solution for the potential and then spend some time justifying it.

Suppose that the reflection amplitude

$$R(\omega) = \frac{\phi_r(\omega,\tau=0)}{\phi_i(\omega,\tau=0)} \tag{C-1}$$

has been measured at all $\omega$. Take the Fourier transform of this quantity to obtain the time function

$$R(t) = \frac{1}{2\pi} \int d\omega \, e^{-i\omega t} \, R(\omega) \tag{C-2}$$

We then solve for a "Kernel" function $K(t,t')$ via the Gelfand-Levitan equation:

$$K(t,t') = -R(t+t') - \int_{-t}^{t} dt'' \, R(t'+t'') \, K(t,t'') \tag{C-3}$$

($K$ is only needed for $t' \leq t$.) Once $K$ has been found, the potential $V(\tau)$ is just

$$V(\tau) = 2 \frac{d}{d\tau} K(\tau,\tau) \tag{C-4}$$

The justification of this procedure is fairly involved, and can carry the conscientious reader on a real scavenger hunt through the literature. Presented here is enough of the basic theory to plausify the solution while pointing the fanatical reader toward more complete discussions of certain issues.

Our "proof" of the Gelfand-Levitan inversion procedure will cover the following points:

(1) A function looking suspiciously like $K(t,t')$ (mainly because the same symbol is used) is shown to be the kernel of a transformation between the plane-wave solutions of a potential-free Schroedinger equation and the solutions of the full Schroedinger equation. This kernel satisfies equation (C-4) $[2 \frac{d}{d\tau} K(\tau,\tau) = V(\tau)]$.

(2) This kernel function is shown to obey a Gelfand-Levitan-type equation as a direct consequence of its triangularity.

(3) By calculating the norm of the transformation involving K, we show that the Gelfand-Levitan equation satisfied by K is indeed (C-3). The reader is invited to convince himself that only one solution to (C-3) is possible.

We will begin by assuming that the potential $V(\tau)$ is zero at $\tau < 0$ and $\tau > \tau_f$. The more general case will be treated later.

To show that K is the kernel of an integral transformation we must first find the transformation. The potential-free Schroedinger equation

$$\left(\frac{\partial^2}{\partial\tau^2} + \omega^2\right) \phi_0 = 0 \tag{C-5a}$$

has the plane-wave solutions

$$\phi_0^{\pm} = e^{\pm i\omega\tau} \tag{C-5b}$$

From the plane waves we can construct solutions to the full Schroedinger equation (C-2) using a triangular form of the Lippman-Schwinger equation. We begin with a special Green's function

$$g_t(\omega,\tau) = 0 \qquad \tau < 0$$

$$= \frac{\sin \omega \tau}{\omega} \qquad \tau > 0 \tag{C-6a}$$

which is a solution to

$$\left[\frac{\partial^2}{\partial\tau^2} + \omega^2\right] g_t(\omega,\tau-\tau') = \delta(\tau-\tau') \tag{C-6b}$$

We now can define two new wave functions as

$$\phi^{\pm}(\omega,\tau) = \phi_0^{\pm}(\omega,\tau) + \int_0^{\tau} d\tau' \frac{\sin\omega(\tau-\tau')}{\omega} V(\tau')\, \phi^{\pm}(\omega,\tau') \tag{C-7}$$

It is easily verified that $\phi^{\pm}$ are solutions of the full Schroedinger equation (B-2), so we have defined a transformation linking $\phi_0^{\pm}$ with $\phi^{\pm}$ for all $\omega$. The functions $\phi^{\pm}$ have the nice property of collapsing to the plane waves $\phi_0^{\pm}$ for $\tau < 0$.

We are now ready to show that the mapping from $\phi_0^{\pm}$ to $\phi^{\pm}$ can be written in the form

$$\phi^{\pm}(\omega,\tau) = \phi_0^{\pm}(\omega,\tau) + \int_{-\tau}^{\tau} K(\tau,\tau')\, \phi_0^{\pm}(\omega,\tau')\, d\tau' \tag{C-8}$$

[The integral kernel $K(\tau,\tau')$ defines a ***triangular*** operator, so called because $K(\tau,\tau')$ is zero for $\tau' > \tau$. Equation (C-8) is a ***Volterra*** equation, because it defines an operator which is the sum of the unit operator and a triangular operator.]

To demonstrate that (C-8) is valid, we just plug it into (C-7) and derive an equation for $K$. We get

$$\int_{-\tau}^{\tau} d\tau'\, \phi_0^{\pm}(\omega,\tau')\, K(\tau,\tau') = \int_0^{\tau} d\tau' \frac{\sin\omega(\tau-\tau')}{\omega} \phi_0^{\pm}(\omega,\tau')\, V(\tau') + \int_0^{\tau} d\tau' \int_{-\tau'}^{\tau'} d\tau'' \frac{\sin\omega(\tau-\tau')}{\omega} \times \phi_0^{\pm}(\omega,\tau'')\, V(\tau')\, K(\tau',\tau'') \tag{C-9}$$

The right-hand side of (C-9) is a bit of a mess. However, if it can be put in the same form as the left-hand side, namely

$$\int_{-\tau}^{\tau} d\tau'\, \phi_0^{\pm}(\omega,\tau')\, Q(\tau,\tau')$$

where Q is some (integral) expression independent of $\omega$, then the completeness of the $\phi_0^{\pm}$ would allow us to identify K with Q. [If, of course, the right-hand side of (C-9) cannot be put in this form, then (C-8) is not valid.]

We will start with the first right-hand-side (R.H.S.) term [call it $I_1^{\pm}(\omega,\tau)$]. Note that (remember $\phi_0^{\pm} = e^{\pm i\omega\tau}$)

$$\frac{\sin\omega(\tau-\tau')}{\omega}\,\phi_0^{\pm}(\omega,\tau') = \frac{1}{2}\int_{2\tau'-\tau}^{\tau} d\tau''\,\phi_0^{\pm}(\omega,\tau'')$$

So the first RHS term in (C-9) may be written as

$$I_1^{\pm}(\omega,\tau) = \frac{1}{2}\int_0^{\tau} d\tau'\,V(\tau')\int_{2\tau'-\tau}^{\tau} d\tau''\,\phi_0^{\pm}(\omega,\tau'')$$

Blithely changing the order of integration, we get

$$I_1^{\pm}(\omega,\tau) = \frac{1}{2}\int_{-\tau}^{\tau} d\tau''\,\phi_0^{\pm}(\omega,\tau'')\int_0^{(\tau+\tau'')/2} d\tau'\,V(\tau') \qquad \text{(C-10a)}$$

which is the required form.

For the second RHS term in (C-9) [call it $I_2^{\pm}(\omega,\tau)$] we can use the identity

$$\frac{\sin\omega(\tau-\tau')}{\omega}\,\phi_0^{\pm}(\omega,\tau'') = \frac{1}{2}\int_{\tau''-\tau+\tau'}^{\tau''-\tau'+\tau} d\tau'''\,\phi_0^{\pm}(\omega,\tau''')$$

which yields

$$I_2^{\pm}(\omega,\tau) = \frac{1}{2}\int_0^{\tau} d\tau'\,V(\tau')\int_{-\tau'}^{\tau'} d\tau''\,K(\tau',\tau'')\int_{\tau''-\tau+\tau'}^{\tau''-\tau'+\tau} d\tau'''\,\phi_0^{\pm}(\omega,\tau''')$$

We again need to change the order of integration. It is convenient to define two new integration variables

$$\mu = \frac{\tau'+\tau''}{2}, \qquad \nu = \frac{\tau'-\tau''}{2}$$

(This is not essential, but it simplifies the limits of the integrals.) Then

$$I_2^{\pm}(\omega,\tau) = \int_{-\tau}^{\tau} d\tau''' \, \phi_0^{\pm}(\omega,\tau''') \int_0^{(\tau+\tau''')/2} d\mu \cdot$$

$$\cdot \int_0^{(\tau-\tau''')/2} d\nu \, V(\mu+\nu) \, K(\mu+\nu,\mu-\nu) \qquad \text{(C-10b)}$$

which is also of the required form. Thus, we have as an expression for $K(\tau,\tau')$ when $\tau' \in (-\tau,\tau)$

$$K(\tau,\tau') = \frac{1}{2} \int_0^{(\tau+\tau')/2} d\tau'' \, V(\tau'')$$

$$+ \int_0^{(\tau+\tau')/2} d\mu \int_0^{(\tau-\tau')/2} d\nu \, V(\mu+\nu) \, K(\mu+\nu,\mu-\nu) \qquad \text{(C-11)}$$

This equation establishes the consistency of (C-8) with (C-7). Note that as $\tau' \to \tau$, the second integral in (C-11) $\to 0$, giving the expression

$$K(\tau,\tau) = \frac{1}{2} \int_0^{\tau} d\tau' \, V(\tau') \qquad \text{(C-12)}$$

or,

$$2 \frac{d}{d\tau} K(\tau,\tau) = V(\tau)$$

which is just equation (C-4). (The interested reader can note that the above derivation is going to work only if $V$ is of the Schroedinger form; that is, independent of $\omega$ and local in $\tau$.)

We have yet to establish that the $K(\tau,\tau')$ defined by equation (C-8) is indeed the solution of the Gelfand-Levitan equation (C-3). To do this will require a couple of observations. First is that the triangular nature of $K$ dictates that it satisfy a Gelfand-Levitan-type equation. Second is that the kernel of the Gelfand-Levitan equation for $K$ is in fact the reflection response R. Both observations will require that we examine the properties of the mapping $\phi_0^{\pm} \to \phi^{\pm}$.

If this mapping were unitary, then $\phi^{\pm}$ would have the same normalization, orthogonality, and completeness properties as $\phi_0^{\pm}$. There is, however, no reason why this should be the case, and we are forced to conclude that the mapping is probably not unitary. We can state that $\phi^{\pm}(\omega)$ is orthogonal to $\phi^{\pm}(\omega')$ if $\omega' \neq \omega$, since $\phi^{\pm}(\omega)$ are eigenfunctions of a Hermitian operator with eigenvalue $\omega$. $\phi^{+}(\omega)$ and $\phi^{-}(\omega)$ are definitely linearly independent (by construction), even though they may not be orthogonal. The most general ortho-normalization for the $\phi^{\pm}$ is

$$\langle\phi^{s}(\omega)|\phi^{s'}(\omega')\rangle \equiv \int d\tau\ \phi^{s}(\omega,\tau)^{*}\ \phi^{s'}(\omega',\tau)$$

$$= 2\pi\ \delta(\omega-\omega')\ M_{ss'}(\omega) \qquad \text{(C-13)}$$

[The s indicates + or -, and the $2\pi$ is there just to make this equation look more like the normalization equation $\langle\phi_0^{s}(\omega)|\phi_0^{s'}(\omega')\rangle = 2\pi\ \delta(\omega-\omega')\ \delta_{ss'}$ for $\phi_0$.] $M_{ss'}$ defines, for a given frequency, a little 2 x 2 matrix

$$\underset{\sim}{M}(\omega) = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \qquad \text{(C-14)}$$

which, because of the linear independence of $\phi^{+}$ to $\phi^{-}$, must be invertible.

If the $\phi^{\pm}$ form a complete set (for our purposes this means that any solution to the Schroedinger equation be expressible as a linear combination of them; this will be true unless the potential has "bound state" solutions) then we can also write a completeness relation

$$\hat{1} = \frac{1}{2\pi} \sum_{s,s'=+}^{-} \int M^{-1}_{s's}|\phi^{s}(\omega)\rangle\langle\phi^{s'}(\omega)|d\omega \qquad \text{(C-15a)}$$

abstractly, or, in terms of the wave functions,

$$\delta(\tau-\tau') = \frac{1}{2\pi} \sum_{s,s'=+}^{-} \int d\omega\ M^{-1}_{s',s}\ \phi^{s}(\omega,\tau)\ \phi^{s'}(\omega,\tau')^{*} \qquad \text{(C-15b)}$$

If the $\phi^{\pm}$ are not complete, then the unit operator or delta function in (C-15) becomes a projector onto the space spanned by the $\phi^{\pm}$. In any case, it is trivial to confirm, using (C-13), that the operator defined by (C-15) acts as the unit operator on any linear combination of the $\phi^{\pm}$.

It may not look like it yet, but equation (C-15) is the Gelfand-Levitan equation. It will take a couple of steps to make that obvious. Define the mapping from $\phi_0^{\pm}$ to $\phi^{\pm}$ abstractly as the operator U. In Dirac notation,

$$|\phi^{s}(\omega)\rangle = U|\phi_0^{s}(\omega)\rangle \tag{C-16a}$$

According to (C-8), U may be written as

$$U = I + K \tag{C-16b}$$

where I is the unit operator and K is the triangular operator with elements $K(\tau,\tau')$. Equation (C-15) can be written as

$$\hat{1} = \frac{1}{2\pi}\sum_{s,s'}\int d\omega\, M^{-1}_{s's}(\omega)\; U|\phi_0^{s}(\omega)\rangle\langle\phi_0^{s'}(\omega)|U^{\dagger}$$

$$= U\,W\,U^{\dagger} \tag{C-17}$$

where the operator W is defined as

$$W = \frac{1}{2\pi}\sum_{s,s'}\int d\omega\, M^{-1}_{s's}(\omega)\; |\phi_0^{s}(\omega)\rangle\langle\phi_0^{s'}(\omega)| \tag{C-18a}$$

The elements of this operator equation are

$$W(\tau,\tau') = \frac{1}{2\pi}\sum_{s,s'}\int d\omega\, M^{-1}_{s's}(\omega)\; \phi_0^{s}(\omega,\tau)\; \phi_0^{s'}(\omega,\tau')^{*} \tag{C-18b}$$

$$= \frac{1}{2\pi}\sum_{s,s'}\int d\omega\, M^{-1}_{s's}(\omega)\; e^{i\omega(s\tau-s'\tau')} \tag{C-18c}$$

Equation (C-17) may be written as

$$U\,W = U^{\dagger-1} \tag{C-19a}$$

Now, if $U = I + K$, where $K$ is triangular, it is easy enough to verify that $U^{-1}$ has the same form. Not only that, but $U^{\dagger-1}$ (being the complex conjugate transpose of $U^{-1}$) is also of the same form, except that its elements are triangular in the opposite sense. That is, if we write

$$U^{\dagger-1} = I + L$$

then

$$L(\tau,\tau') = 0 \qquad \text{if} \quad \tau > \tau'$$

If we now write

$$W = I + \Omega \tag{C-18d}$$

(this just defines an operator $\Omega$), then equation (C-19a) takes the form

$$K + \Omega + K\Omega = L \tag{C-19b}$$

or, as an integral equation

$$K(\tau,\tau') + \Omega(\tau,\tau') + \int_{-\tau}^{\tau} d\tau'' \, K(\tau,\tau'') \, \Omega(\tau'',\tau') = L(\tau,\tau') \tag{C-19c}$$

Since $L(\tau,\tau') = 0$ for $\tau' > \tau$, this is a Gelfand-Levitan equation.

We have just seen that the Gelfand-Levitan equation is just the completeness equation for a Volterra operator. It remains to be seen that the operator $\Omega = W - I$ in equation (C-19c) is the reflection function appearing in (C-3).

This means we will actually have to figure out what the normalization matrix $M$ is. To do this is not so hard as one might imagine, though we do need, for every $\omega$, to consider two experiments. In one, an incident wave $\phi_0^+(\omega,\tau)$ is incident from above, producing a reflected wave $R(\omega)\,\phi_0^-(\omega,\tau)$ for $\tau < 0$ and a transmitted wave $T(\omega)\,\phi_0^+(\omega,\tau)$ in the region $\tau > \tau_f$. This was the actual experiment, $R(\omega)$ the actual data. The other experiment has a plane wave $\phi_0^-(\omega,\tau)$ coming in from the region $\tau > \tau_f$, a reflected wave $\tilde{R}(\omega)\ \phi_0^+(\omega,\tau)$ in that region and a transmitted wave $\tilde{T}(\omega)\ \phi_0^-(\omega,\tau)$ in the region $\tau < 0$. This experiment was never done owing to certain technical problems, but we can still think about it.

These two experiments generate two solutions $\psi_\pm(\omega,\tau)$ of the full Schroedinger equation. We have

$$\phi_0^+(\omega,\tau) + R(\omega)\ \phi_0^-(\omega,\tau) \underset{\tau<0}{\leftarrow} \psi_+(\omega,\tau) \underset{\tau>\tau_f}{\rightarrow} T(\omega)\ \phi_0^+(\omega,\tau) \qquad \text{(C-20a)}$$

$$\tilde{T}(\omega)\ \phi_0^-(\omega,\tau) \underset{\tau<0}{\leftarrow} \psi_-(\omega,\tau) \underset{\tau>\tau_f}{\rightarrow} \phi_0^-(\omega,\tau) + \tilde{R}(\omega)\ \phi_0^+(\omega,\tau) \qquad \text{(C-20b)}$$

Since the Wronskian of any two solutions of the Schroedinger equation is independent of $\tau$, it is easily established that the four quantities $R$, $\tilde{R}$, $T$, $\tilde{T}$ are not independent. We have in fact[1]

$$\tilde{T}(\omega) = T(\omega) \qquad \text{(C-21a)}$$

$$T(\omega)^*\tilde{R}(\omega) = -R(\omega)^*T(\omega) \qquad \text{(C-21b)}$$

$$|R(\omega)|^2 + |T(\omega)|^2 = 1 \qquad \text{(C-21c)}$$

Nevertheless, the two solutions $\psi_+$ and $\psi_-$ are independent, and are in fact orthogonal. We have as normalization

$$\langle\psi_s(\omega)|\psi_{s'}(\omega')\rangle = 2\pi\ \delta(\omega-\omega')\ \delta_{ss'} \qquad \text{(C-22)}$$

This may be proved by noting that $\psi_\pm$ are in fact the solutions of the Lippmann-Schwinger equations

$$|\psi_\pm(\omega)\rangle = |\phi_0^\pm(\omega)\rangle + G_0 V|\psi_\pm(\omega)\rangle \qquad \text{(C-23)}$$

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[1]To derive these, you need to realize that the complex conjugates of $\psi_\pm$ are also solutions to the Schroedinger equation.

where $G_0$ is the exploding Green's function for the potential-free Schroedinger equation, and that the Lippman-Schwinger equation (C-23) defines a unitary transformation. [If this is not obvious, hold on. More will be said when we generalize the limits of $V(\tau)$.]

So, if the solutions $\phi^{\pm}$ can be expressed in terms of the $\psi_{\pm}$, the little normalization matrix $\underset{\sim}{M}$ for the $\phi^{\pm}$ is easily computed.

From the asymptotic form

$$\phi^{\pm}(\omega,\tau) \underset{\tau<0}{\rightarrow} \phi_0^{\pm}(\omega,\tau)$$

we have

$$\psi_+(\omega,\tau) = \phi^+(\omega,\tau) + R(\omega)\, \phi^-(\omega,\tau)$$

$$\psi_-(\omega,\tau) = \tilde{T}(\omega)\, \phi^-(\omega,\tau)$$

or, solving for $\phi^{\pm}$,

$$\phi^+(\omega,\tau) = \psi_+(\omega,\tau) - \frac{R(\omega)}{\tilde{T}(\omega)} \psi_-(\omega,\tau) \qquad \text{(C-24a)}$$

$$\phi^-(\omega,\tau) = \frac{1}{\tilde{T}(\omega)} \psi_-(\omega,\tau) \qquad \text{(C-24b)}$$

With the normalization (C-22) for $\psi_{\pm}$, this gives $\underset{\sim}{M}$ as

$$\underset{\sim}{M}(\omega) = \frac{1}{|\tilde{T}(\omega)|^2} \begin{pmatrix} |\tilde{T}(\omega)|^2 + |R(\omega)|^2 & -R(\omega)^* \\ -R(\omega) & 1 \end{pmatrix}$$

or, with the relations (C-21) between $R$ and $\tilde{T}$,

$$\underset{\sim}{M}(\omega) = \frac{1}{1 - |R(\omega)|^2} \begin{pmatrix} 1 & -R(\omega)^* \\ -R(\omega) & 1 \end{pmatrix} \qquad \text{(C-25)}$$

The inverse of $\underset{\sim}{M}$ then has the form

$$\underset{\sim}{M}^{-1}(\omega) = \begin{pmatrix} 1 & R(\omega)^* \\ R(\omega) & 1 \end{pmatrix} \tag{C-26}$$

which gives for the weight function W defined in (28)

$$\begin{aligned} W(\tau,\tau') &= \frac{1}{2\pi} \sum_{s,s'=\pm} \int d\omega\, M^{-1}_{s's}(\omega)\, e^{i\omega(s\tau - s'\tau')} \\ &= \frac{1}{2\pi} \int_0^\infty d\omega \left[ e^{i\omega(\tau-\tau')} + R(\omega)\, e^{-i\omega(\tau+\tau')} \right] + \text{c.c.} \\ &= \delta(\tau-\tau') + R(\tau+\tau') \end{aligned}$$

The function $\Omega(\tau,\tau')$ appearing in (C-19) is just $R(\tau + \tau')$, so (C-19) is the same Gelfand-Levitan equation as (C-3).

To completely justify the Gelfand-Levitan procedure we should prove that there is at most one solution to (C-3), and moreover demonstrate convergence of a few integrals used along the way. These details will be left to the fanatical reader.

## *D. Potentials with Two Asymptotic Values*

The Gelfand-Levitan algorithm developed in part C required a potential which was zero for $\tau$ outside the interval $(0,\tau_f)$. Actually, the same algorithm will work for potentials which approach any constant value as $\tau \to \infty$, provided the approach is sufficiently rapid.

In part B we showed that for the seismic problem, post-critical angles of incidence may be handled by converting the wave equation to a Schroedinger equation whose potential approaches a different value at large $\tau$ than at small $\tau$. We will now extend the Gelfand-Levitan algorithm to this case.

Suppose

$$V(\tau) = V_0 \qquad \tau \le 0$$

$$= V_f \qquad \tau \ge \tau_f$$

Normally, since velocity increases with depth, $V_f > V_0$.

We may, in this case, still devise scattering experiments. For all frequencies greater than $(V_0)^{\frac{1}{2}}$ a wave may be sent into the earth from above, producing a reflected wave in the region $\tau < 0$ and a transmission into the region $\tau > \tau_f$. Thus a solution to the full Schroedinger equation is $\psi_+(\omega,\tau)$, with asymptotes

$$e^{i\omega_0\tau} + R(\omega)\, e^{-i\omega_0\tau} \underset{\tau<0}{\leftarrow} \psi_+(\omega,\tau) \underset{\tau>\tau_f}{\rightarrow} T(\omega)\, e^{i\omega_f\tau} \tag{D-2a}$$

Here we have used reduced frequencies $\omega_0$ and $\omega_f$ defined as

$$\omega_0 = \sqrt{\omega^2 - V_0} \tag{D-3a}$$

$$\omega_f = \sqrt{\omega^2 - V_f} \qquad \omega^2 > V_f$$

$$= i\sqrt{V_f - \omega^2} \qquad \omega^2 < V_f \tag{D-3b}$$

reflecting the fact that for $\tau < 0$, $\psi$ obeys the Schroedinger equation

$$(\partial_\tau^2 + \omega^2 - V_0)\,\psi = 0 \qquad \tau < 0 \tag{D-3c}$$

while for $\tau > \tau_f$, it obeys another:

$$(\partial_\tau^2 + \omega^2 - V_f)\,\psi = 0 \qquad \tau > \tau_f \tag{D-3d}$$

We may also think of another experiment in which plane waves are incident from below, producing for $\omega > (V_f)^{\frac{1}{2}}$ another function $\psi_-$:

$$\tilde{T}(\omega)\, e^{-i\omega_0\tau} \underset{\tau<0}{\leftarrow} \psi_-(\omega,\tau) \underset{\tau>\tau_f}{\rightarrow} e^{-i\omega_f\tau} + \tilde{R}(\omega)\, e^{i\omega_f\tau} \qquad \text{(D-2b)}$$

Together, $\psi_+$ and $\psi_-$ constitute all the scattering experiments that can be performed. This gives a hint about the difficulties we are about to encounter. For the simpler case in section C, in order to solve for $V(\tau)$, it was necessary to generate at least one solution to the Schroedinger equation at every frequency. The lowest frequency we can generate from above is $\omega = (V_0)^{\frac{1}{2}}$, so if our Schroedinger equation has solutions for $\omega < (V_0)^{\frac{1}{2}}$ (which will happen if $V(\tau)$ drops significantly below $V_0$ over a large enough $\tau$-interval) we are likely in trouble. If $V_0 > V_f$, we are definitely in trouble. But enough of that for now.

By comparing the Wronskians of the asymptotes of $\psi_\pm$, the following relations may be derived:

$$|R(\omega)|^2 + \frac{\omega_f}{\omega_0}|T(\omega)|^2 = 1 \qquad \omega^2 > V_f \qquad \text{(D-4a)}$$

$$|R(\omega)|^2 = 1 \qquad V_0 < \omega^2 < V_f \qquad \text{(D-4b)}$$

$$\tilde{T}(\omega) = \frac{\omega_f}{\omega_0} T(\omega) \qquad \omega^2 > V_f \qquad \text{(D-4c)}$$

$$|\tilde{R}(\omega)|^2 + \frac{\omega_0}{\omega_f}|\tilde{T}(\omega)|^2 = 1 \qquad \omega^2 > V_f \qquad \text{(D-4d)}$$

$$R(\omega) = -\tilde{R}(\omega)^* T(\omega)/T(\omega)^* \qquad \omega^2 > V_f \qquad \text{(D-4e)}$$

These relations have been written down assuming $V_0 < V_f$.

We will now assume that $\psi_+$ and $\psi_-$ together form a complete set of solutions to the Schroedinger equation [which is again tantamount to requiring that $V_0$ (or $V_f$) be essentially the lowest potential at any depth].

The normalization of the $\psi_\pm$ is expressible as

$$\langle\psi_+(\omega)|\psi_+(\omega')\rangle = 2\pi\,\delta(\omega_0-\omega_0') \qquad \text{(D-5a)}$$

$$\langle\psi_+(\omega)|\psi_-(\omega')\rangle = 0 \qquad \text{(D-5b)}$$

$$\langle\psi_-(\omega)|\psi_-(\omega')\rangle = 2\pi\,\delta(\omega_f-\omega_{f'}) \qquad \text{(D-5c)}$$

These relations are easily proved by noting that $\psi_+$ comes from a norm-preserving transformation of $\exp(i\omega_0\tau)$ and $\psi_-$ comes from a norm-preserving transformation of $\exp(-\omega_f\tau)$ [see Kay and Moses (1955), Appendix II, for a guideline]. So, if $\psi_+$ and $\psi_-$ together are complete, we have

$$2\pi\,\delta(\tau-\tau') = \int_0^\infty d\omega_0\; \psi_+(\omega,\tau)\;\psi_+(\omega,\tau')^* + \int_0^\infty d\omega_f\; \psi_-(\omega,\tau)\;\psi_-(\omega,\tau')^* \qquad \text{(D-6)}$$

Note that the integration variables in these two integrals are $\omega_0$ and $\omega_f$ rather than $\omega$.

So where do we go from here? Solutions $\phi^\pm(\omega,\tau)$ analagous to those in (C-7) can now be defined:

$$\phi^\pm(\omega,\tau) = e^{\pm i\omega_0\tau} + \int_0^\tau d\tau'\;\frac{\sin\omega(\tau-\tau')}{\omega}\,[V(\tau') - V_0]\;\phi^\pm(\omega,\tau') \qquad \text{(D-7)}$$

Since $\phi^\pm$ have the asymptotic form $\phi^\pm \underset{\tau<0}{\rightarrow} e^{\pm i\omega_0\tau}$, $\psi_\pm$ can be expressed in terms of $\phi^\pm$ as

$$\psi_+(\omega,\tau) = \phi^+(\omega,\tau) + R(\omega)\;\phi^-(\omega,\tau) \qquad \text{(D-8a)}$$

$$\psi_-(\omega,\tau) = \tilde{T}(\omega)\;\phi^-(\omega,\tau) \qquad \text{(D-8b)}$$

The completeness relation (D-6) can then be put in terms of $\phi^\pm$:

$$2\pi\,\delta(\tau-\tau') = \int_0^\infty d\omega_0\;[\phi^+(\omega,\tau)\;\phi^+(\omega,\tau')^* + R(\omega)\;\phi^-(\omega,\tau)\;\phi^+(\omega,\tau')^* +$$

$$+ R^*(\omega)\ \phi^+(\omega,\tau)\ \phi^-(\omega,\tau')^*$$

$$+ |R(\omega)|^2\ \phi^-(\omega,\tau)\ \phi^-(\omega,\tau')^*]$$

$$+ \int_0^\infty d\omega_f\ |\tilde{T}(\omega)|^2\ \phi^-(\omega,\tau)\ \phi^-(\omega,\tau')^* \qquad \text{(D-9)}$$

If we take $\phi_0^\pm(\omega,\tau)$ to be $e^{\pm i\omega_0\tau}$, then the Volterra transformation (C-8) linking $\phi_0$ to $\phi$ can be defined. Abstractly, we can write

$$|\phi^\pm(\omega)|> \;=\; (I+K)|\phi_0^\pm(\omega)> \;\equiv\; U|\phi_0^\pm(\omega)> \qquad \text{(D-10)}$$

where $K$ is the triangular operator with the property

$$2\frac{d}{d\tau}K(\tau,\tau) \;=\; V(\tau) - V_0 \qquad \text{(D-11)}$$

This allows us to write (D-9) in the abstract form

$$\hat{1} \;=\; U\,W\,U^+ \qquad \text{(D-12)}$$

where

$$2\pi\,W \;=\; \int_0^\infty d\omega_0[\ \phi_0^+(\omega)><\phi_0^+(\omega)| + R(\omega)\ |\phi^-(\omega)><\phi^+(\omega)|$$

$$+ R^*(\omega)\ |\phi^+(\omega)|><\phi^-(\omega)| + |R(\omega)|^2|\phi^-(\omega)><\phi^-(\omega)|]$$

$$+ \int_0^\infty d\omega_f\ |\tilde{T}(\omega)|^2\ |\phi^-(\omega)><\phi^-(\omega)| \qquad \text{(D-13)}$$

Using the relation $d\omega_f = d\omega_0\ \omega_0/\omega_f$, plus equations (D-4c) and (D-4a), the second integral in (D-13) can be rewritten as

$$\int_{(V_f-V_0)^{\frac{1}{2}}}^\infty d\omega_0\ [1 - |R(\omega)|^2]\ |\phi^-(\omega)><\phi^-(\omega)|$$

The lower limit of this integral may be extended to zero since, according to equation (D-4b), $1 - |R(\omega)|^2 = 0$ for $\omega \in [0,(V_f-V_0)^{1/2}]$. (D-13) therefore can be written as

$$W = \frac{1}{2\pi}\int_0^\infty d\omega_0[\phi_0^+(\omega)\rangle\langle\phi_0^+(\omega)| + |\phi_0^-(\omega)\rangle\langle\phi_0^-(\omega)|$$

$$R(\omega)\ |\phi^-(\omega)\rangle\langle\phi^+(\omega)| + R^*(\omega)\ |\phi^+(\omega)\rangle\langle\phi^-(\omega)|]$$

or, in the $\tau$-representation,

$$W(\tau,\tau') = \frac{1}{2\pi}\int_0^\infty d\omega_0[e^{i\omega_0(\tau-\tau')} + d^{-i\omega_0(\tau-\tau')}$$

$$+ R\, e^{-i\omega_0(\tau+\tau')} + R^* e^{i\omega_0(\tau+\tau')}]$$

$$= \delta(\tau-\tau') + R(\tau+\tau')$$

which is exactly the same weight function as was derived in section C. This leads us to the same Gelfand-Levitan equation (C-3) as before. The only change made to the Gelfand-Levitan algorithm by the more general potential is the replacement of equation (C-4) by (D-11) (big deal).

If the potential $V(\tau)$ has bound states, or if we conduct the scattering experiments from the high-velocity side, the effect on the weight function (D-14) will be to add a term which depends on parameters which we haven't measured. Thus only if $V_0$ is essentially the lowest potential at any depth is a unique inverse obtainable. Even then, we may be in a spot of trouble. If the potential has a "well" or low spot in any region, even if the bottom of the low spot has a potential higher than $V_0$, it can be illuminated completely only by wave functions which are evanescent (exponentially decaying) in the high-potential regions. This is bound to cause trouble in any practical computation.

## *E. Relating an Impulse Response to the Plane-Wave Problem*

In the actual seismic experiment one deals with a filtered impulse response rather than with plane waves. We can, however, establish a correspondence, the details of which depend on the experiment.

As a simple example suppose a free surface exists at $z=0$, and that velocity is constant down at least to depth $z=a$, where we place an impulsive source and a string of receivers. In the $\tau$-coordinate system, we wish to convert the impulse response $G(\omega;\tau_r=\tau_a|\tau_s=\tau_a)$ $(\tau_a=a/v_0)$ into the plane-wave reflection coefficient $R(\omega)$.

This is easy enough to do. At any receiver depth $\tau_r$ and source depth $\tau_s$, we can express $G$ as

$$G(\omega;\tau_r|\tau_s) = J(\omega)\ \chi(\omega,\tau_<)\ \psi_+(\omega,\tau_>) \qquad \text{(E-1)}$$

where $\tau_<$ $(\tau_>)$ signifies the smaller (larger) of $\tau_r$, $\tau_s$ . $\psi_+$ is the solution to the plane-wave experiment, and $\chi$ is a solution of the Schroedinger equation which goes to zero at $\tau = 0$. $J(\omega)$ is there to give the impulse response the proper normalization, namely

$$\left[\frac{\partial^2}{\partial\tau_r^2} + \omega^2 - V(\tau_r)\right] G(\omega,\tau_r|\tau_s) = \delta(\tau-\tau') \qquad \text{(E-2)}$$

Since we only need $G$ in the zero-potential region, it will be very easy to construct.

A solution $\chi(\omega,\tau)$, which goes to zero at the origin, is

$$\chi(\omega,\tau) = \sin\omega\tau$$

Using the asymptotic form

$$\psi_+ = e^{i\omega\tau} + R(\omega)\ e^{-i\omega\tau}$$

the choice for $J(\omega)$ which gives the proper normalization as

$$J(\omega) = \frac{-1}{\omega[1 + R(\omega)]}$$

or

$$G(\omega;\tau_a|\tau_a) = -\frac{\sin \omega \tau_a}{\omega} \frac{e^{i\omega\tau_a} + R(\omega)\, e^{-i\omega\tau_a}}{1 + R(\omega)} \qquad \text{(E-3)}$$

If $G$ is the measured quantity, then $R(\omega)$ is just

$$R(\omega) = -\frac{G(\omega;\tau_a|\tau_a) + \frac{\sin \omega \tau_a}{\omega} e^{i\omega\tau_a}}{G(\omega;\tau_a|\tau_a) + \frac{\sin \omega \tau_a}{\omega} e^{-i\omega\tau_a}} \qquad \text{(E-4)}$$

If the measured quantity is $G$ less a direct arrival $G_o$, we have as data

$$D(\omega) = G(\omega;\tau_a|\tau_a) - G_o(\omega;\tau_a|\tau_a)$$

where

$$G_o = -\frac{\sin \omega \tau_a}{\omega} e^{i\omega\tau_a}$$

An expression for $R$ in terms of the measured data field is

$$R(\omega) = -\frac{D(\omega)}{D(\omega) - \frac{2i \sin^2 \omega\tau_a}{\omega}} \qquad \text{(E-5)}$$

Expressions similar to (E-5), tailored to a particular seismic experiment, should be easily derivable.

## F. *Conclusions*

The Gelfand-Levitan method seems applicable to the inverse seismic problem in a layered medium. It does require that the reflection $R(\omega)$ be determinable at all frequencies. This can't be done, really. A practical source will be band-limited, its phase and amplitude characteristics poorly known at best. The effects of these imperfections on the Gelfand-Levitan inverse is not crystal clear, but a good case, even for a layered medium, could be made for an iterative inverse method which allows one to estimate unknown amplitudes and phases as one goes.

Nevertheless, my conclusion is that if anyone ever discovers a layered medium, we will be ready for it.

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