AN INVERSION METHOD FOR ELASTIC WAVE FIELDS

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Abstract

The inversion of two-dimensional elastic displacement fields can be handled in a very similar manner to the way the acoustic problem is handled. The Born approximation of the Lippman-Schwinger equation yields a simple relationship in the Fourier-transform domain between the observed horizontal and vertical displacement fields, and the scattering potential. Basically, the observations are a linear combination of the scattering potential evaluated along four different shells. The four shells may be interpreted P to P, P to S, S to P, and S to S scattering.

If the source is either purely compressional or purely shear, then one experiment will suffice to invert the forward equation. If the source is a (known) mixture of P and S components, then two experiments with different combinations of P and S components are necessary for the inversion.

Introduction

In the paper "An Inversion Method for Acoustic Wave Fields", the Born approximation was used to relate the "reflectivity" function to the density and bulk-modulus variations. In this paper, we apply the same approach to the two-dimensional elastic problem. In this case there is a substantial advantage in determining the form of the reflectivity because there are four reflectivity functions, but only three medium parameters.

The field experiment necessary for the inversion method is a standard multi-offset reflection survey with two components of displacement (horizontal and vertical) recorded at each geophone location. It is (apparently) necessary to cast the elastic inversion method in terms of displacements because exact wave operators for variable media can only be cast in terms of these variables.

The use of the Born approximation will force several restrictions on the procedure. Basically, the background P- and S-wave velocities must be constant. The inversion scheme is limited to sub-critical reflections, and it has no provision for handling multiples.

The Forward Scattering Equation

The starting point of the derivation is the two-dimensional elastic displacement equation for a linear isotropic medium

$$L\,\mathbf{u} = (\partial_z A\,\partial_z + \partial_z B\,\partial_x + \partial_x B^T \partial_z + \partial_x C\,\partial_x + \rho\omega^2 I)\,\mathbf{u} = 0 \qquad (1)$$

where

$$A = \begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \mu & 0 \\ 0 & \lambda+2\mu \end{pmatrix}$$

and $\mathbf{u}$ is the displacement vector $(u,w)^T$. This is the form of the operator used in previous SEP reports (cf. Clayton and Brown, SEP-20, pp. 73-96). For the derivation here, it is convenient to rewrite the operator in an equivalent form:

$$L = \nabla \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \nabla^T + 2H \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H^T - 2H^T \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H + \rho\omega^2 I \qquad (2)$$

where

$$\nabla = \begin{pmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{pmatrix} \qquad H = \begin{pmatrix} 0 & \partial_z \\ \partial_x & 0 \end{pmatrix}$$

and $\gamma=\lambda+2\mu$. Note the normalizations for the operators ∇ and H

$$\nabla^T \nabla = \nabla \nabla^T = (\partial_{xx} + \partial_{zz})I = \nabla^2 I \qquad \text{and} \qquad H^T H = H H^T = \partial_x \partial_z I$$

This form of the elastic displacement equation has a number of advantages. First, if the shear modulus is constant, then the terms involving the operator H annihilate each other. The resulting equation is very similar in form to the scalar displacement equation.[1] Second, as will be shown later, the term involving the operator ∇ will give rise to primary scattering (P to P, and S to S), while the terms involving H generate converted scattering (P to S, and S to P). This implies that the converted scattering is primarily governed by the shear modulus.

The operator ∇^T acting on the displacement field produces the divergence and curl of that field. This means that it converts displacements to potentials. The operator ∇ acting on the potential variables produces displacements.

The Born approximation of the Lippman-Schwinger equation is

$$G = G_o + G_o V G_o \tag{3}$$

where $V = L - L_o$. This equation is valid for the elastic case, if we realize that the Green's operators, and the scattering potential are dyadics.

The problem about which we will perturb is the one for which the wave operator is

$$L_o = \rho_o \omega^2 + \nabla \begin{pmatrix} \gamma_o & 0 \\ 0 & \mu_o \end{pmatrix} \nabla^T \tag{4}$$

Hence, the scattering potential is

[1] The scalar equation referred to is the SH displacement equation $(\rho\omega^2 + \nabla\cdot\mu\,\nabla)u = 0$.

$$V = (\rho-\rho_0)\omega^2 I + \nabla \begin{pmatrix} \gamma-\gamma_0 & 0 \\ 0 & \mu-\mu_0 \end{pmatrix} \nabla^T + 2H \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H^T - 2H^T \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H \qquad (5)$$

We will define the dimensionless parameters

$$a = \frac{\gamma}{\gamma_0} - 1, \qquad b = \frac{\mu}{\mu_0} - 1, \quad \text{and} \qquad c = \frac{\rho}{\rho_0} - 1.$$

As with the scalar inversion, we will concentrate on finding the dimensionless functions above, and not worry about reconstructing the actual medium parameters. With the above definitions, the scattering potential becomes

$$V = \rho_0 \left[c\omega^2 I + \nabla \begin{pmatrix} \alpha^2 a & 0 \\ 0 & \beta^2 b \end{pmatrix} \nabla^T + 2\beta^2 H \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} H^T - 2\beta^2 H^T \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} H \right] \qquad (6)$$

where α and β, defined as

$$\alpha^2 = \frac{\gamma_0}{\rho_0} \qquad \text{and} \qquad \beta^2 = \frac{\mu_0}{\rho_0}$$

are the background P- and S-wave velocities.

In this paper, we will not be treating a free surface.[2] Instead, we will stop the medium above the datum from scattering by assuming that $a(x,z)$, $b(x,z)$, and $c(x,z)$ are zero for $z<0$.

For a point source, the observed reflected wave field is related to the scattering potential by

$$\Psi(x_g, x_s, \omega) = G_0 V G_0 F S(\omega) \qquad (7)$$

where F is a two-component vector representing the relative source strengths in u and w. $S(\omega)$ is the transform of the source time function.

[2] This is a more significant assumption in the elastic case because we neglect mode conversion on the free surface.

The Scattering Equation in the Frequency Domain

Equation (7) has a more useful form in the Fourier-transform domain. Transforming over x_g and x_s we have

$$\Psi(k_g,k_s,\omega) = \langle k_g|x_g\rangle\langle x_g,0|G_0|x',z'\rangle \langle x',z'|V|x'',z''\rangle$$

$$\langle x'',z''|G_0|x_s,0\rangle\langle x_s|k_s\rangle FS(\omega) \qquad (8)$$

Substituting directly from Appendix A we have

$$\Psi(k_g,k_s,\omega) = \frac{-1}{2\pi}\,\frac{1}{\rho_0^2\omega^4} \int dx'\int dz'\int dx''\int dz''\; e^{ik_g x'}\left(A_g e^{-i\nu_g|z'|} + B_g e^{-i\eta_g|z'|}\right)$$

$$V(x',z'|x'',z'')\; e^{-ik_s x''}\left(A_s e^{-i\nu_s|z''|} + B_s e^{-i\eta_s|z''|}\right) FS(\omega) \qquad (9)$$

where we have made the following definitions (from Appendix A)

$$\nu = \nu(k_x,\omega) = \left(\frac{\omega^2}{\alpha^2} - k_x^2\right)^{1/2}, \qquad \eta = \eta(k_x,\omega) = \left(\frac{\omega^2}{\beta^2} - k_x^2\right)^{1/2},$$

$$A = A(k_x,\nu) = \frac{1}{2\nu}\begin{pmatrix} k_x & -\nu \\ \nu & k_x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_x & \nu \\ -\nu & k_x \end{pmatrix}, \qquad (10)$$

and

$$B = B(k_x,\eta) = \frac{1}{2\eta}\begin{pmatrix} k_x & -\eta \\ \eta & k_x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k_x & \eta \\ -\eta & k_x \end{pmatrix} \qquad (11)$$

The subscripts g and s in equation (9) identify the horizontal wavenumber (k_g or k_s) to be used in the above definitions. Hence,

$$\nu_g = \nu(k_g,\omega) \qquad \nu_s = \nu(k_s,\omega) \qquad \eta_g = \eta(k_g,\omega) \qquad \eta_s = \eta(k_s,\omega)$$

and

$$A_g = A(k_g, \nu_g) \qquad A_s = A(k_s, \nu_s) \qquad B_g = B(k_g, \eta_g) \qquad B_s = B(k_s, \eta_s)$$

The operator A selects the compressional components from the displacement fields. It accomplishes this basically by converting into potential variables, selecting the P component, and then reconverting to displacements. The A operator applied to a purely shear field produces a zero result. In a similar fashion, the B operator selects the shear component of the displacement field.

Since $V(x',z'|x'',z'')$ is zero for either $z' < 0$ or $z'' < 0$, the absolute signs in equation (9) may be dropped. This allows us to identify each of the terms in equation (9) as a four-dimensional Fourier transform over x', z', x'', and z''. Hence,

$$\Psi(k_g, k_s, \omega) = -\frac{2\pi}{\rho_0^2 \omega^4} \Big[A_g V(k_g, -\nu_g | k_s, \nu_s) A_s + A_g V(k_g, -\nu_g | k_s, \eta_s) B_s$$

$$B_g V(k_g, -\eta_g | k_s, \nu_s) A_s + B_g V(k_g, -\eta_g | k_s, \eta_s) B_s \Big] \; F \; S(\omega) \qquad (12)$$

Thus, the observed data is a linear combination of the scattering potential evaluated along four different hyper-surfaces or "shells". By noting the positions of the A and B operators, one can identify what type of scattering each shell contributes. For example, the first term involves the operators A_g and A_s, which means that it is P to P type scattering. The next three terms in the sum are respectively S to P scattering, P to S scattering, and finally S to S scattering.

Inversion of the Scattering Equation

The next logical step is to substitute the Fourier transform of the scattering potential given in Appendix B into equation (10). However, since the scattering potential is a sum of three terms, and it appears four times in equation (10) with different arguments, we will simplify things first. We will do this by making some assumptions about the nature of the source.

If the source were purely compressional then only two terms in equation (10) would be non-zero

$$\Psi_P(k_g,k_s,\omega) = -\frac{2\pi}{\rho_0\omega^4}\left[A_g V(k_g,-\nu_g|k_s,\nu_s)A_s + B_g V(k_g,-\eta_g|k_s,\nu_s)A_s\right] S(\omega) \quad (13)$$

Ψ_P is a two-component vector containing the horizontal and vertical components of displacement due to a compressional source.

We can further simplify the problem by exploiting the highly structured from of the operators A and B. It is clear form equations (10) and (11) that both A and B have a zero eigenvalue, and that it occurs in opposite positions (the 22-position for A, and the 11-position for B). Premultipling either A or B by the eigenvector that corresponds to its zero eigenvalue will annihilate the operator. The operators (which are the appropriate eigenvectors of A and B)

$$e_P = [k_g,\eta_g]^T \quad (14)$$

and

$$e_S = [-\nu_g,k_g]^T \quad (15)$$

have the properties

$$e_P\cdot B_g = 0 \quad \text{and} \quad e_S\cdot A_g = 0$$

The operators e_P and e_S have, as one might expect, the form of a divergence and a curl operator, respectively. Applying these operators to equation (13), we have

$$\Psi_{PP}(k_g,k_s,\omega) = e_P\cdot\Psi_P = -\frac{2\pi}{\rho_0^2\omega^4}\, e_P\cdot\left[A_g V(k_g,-\nu_g|k_s,\nu_s)A_s\right] S(\omega) \quad (16)$$

and

$$\Psi_{SP}(k_g,k_s,\omega) = e_S \cdot \Psi_P = -\frac{2\pi}{\rho_0^2\omega^4}\, e_S \cdot \left[B_g V(k_g,-\eta_g | k_s,\nu_s) A_s\right] \tag{17}$$

We have now reduced the problem to the same level as was discussed in the paper on acoustic inversion (this report). To proceed from this point one would transform equations (16) and (17) into midpoint-offset coordinates, and make a change of independent variable $k_z = -\nu_g - \nu_s$ for equation (16), and $k_z = -\nu_g - \eta_s$ for equation (17). Then after determining the coefficients of the scattering potential given in Appendix B in the new coordinate systems, one could least squares fit for the unknowns a, b, and c.

If the source were purely shear, then the other two terms in equation (10) would be the ones that are non-zero. The reduction to two scalar problems is similar in this case.

If the source is a mixture of P and S waves, then two experiments will be required to separate the various contributions. For example, if the source has compressional and shear strengths of p_1 and s_1 for the first experiment, and p_2 and s_2 for the second, then the observed wave fields would be

$$\Psi_1 = \left[p_1\left(A_g V A_s + B_g V A_s\right) + s_1\left(A_g V B_s + B_g V B_s\right)\right] S(\omega) \tag{18}$$

$$\Psi_2 = \left[p_2\left(A_g V A_s + B_g V A_s\right) + s_2\left(A_g V B_s + B_g V B_s\right)\right] S(\omega) \tag{19}$$

For brevity we have omitted the constants in equation (10), and the arguments of V and Ψ. By applying the divergence and curl operators we can reduce these equations to

$$e_P \cdot \Psi_1 = \left[p_1\; e_P \cdot A_g V A_s + s_1\; e_P \cdot A_g V B_s\right] S(\omega) \tag{20}$$

$$e_P \cdot \Psi_2 = \left[p_2\; e_P \cdot A_g V A_s + s_2\; e_P \cdot A_g V B_s\right] S(\omega) \tag{21}$$

and a similar set for the e_S operator. Solving for $e_P \cdot A_g V A_s\, S(\omega)$ and $e_P \cdot A_g V B_s\, S(\omega)$ we have

$$e_P \cdot A_g V A_s \; S(\omega) = \frac{s_2 e_P \cdot \Psi_1 - s_1 e_P \cdot \Psi_2}{s_2 p_1 - s_1 p_2} \tag{22}$$

and

$$e_P \cdot A_g V B_s \; S(\omega) = \frac{p_2 e_P \cdot \Psi_1 - p_1 e_P \cdot \Psi_2}{p_2 s_1 - p_1 s_2} \tag{23}$$

As long as $p_1 s_2 \neq p_2 s_1$ then the problem can be reduced to the scalar case.

APPENDIX A: The Green's Operator For A 2-D Elastic Medium

The equation defining the Green's operator for the 2-D elastic case is

$$\rho_0 \left[\omega^2 I + \nabla \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \nabla^T \right] G_0 = -\delta(x-x')\,\delta(z-z') \tag{A1}$$

where ∇ is defined in equation (2). Fourier transforming over x and z in equation (A1) we have

$$\rho_0 \left[\omega^2 I + \tilde{\nabla} \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \tilde{\nabla}^T \right] G_0 = \frac{-1}{2\pi} e^{ik_x x' + ik_z z'} \tag{A2}$$

where

$$\tilde{\nabla} = i \begin{pmatrix} k_x & -k_z \\ k_z & k_x \end{pmatrix}$$

This equation may now be solved for G_0

$$G_0 = \frac{1}{2\pi\rho_0} \; \tilde{\nabla} \begin{bmatrix} \dfrac{1}{\alpha^2 (k_z-\nu)(k_z+\nu)} & 0 \\ 0 & \dfrac{1}{\beta^2 (k_z-\eta)(k_z+\eta)} \end{bmatrix} \tilde{\nabla}^T \; \frac{e^{ik_x x' + ik_z z'}}{k_x^2 + k_z^2} \tag{A3}$$

where

$$\nu = \left(\frac{\omega^2}{\alpha^2} - k_x^2\right)^{\frac{1}{2}} \qquad \text{and} \qquad \eta = \left(\frac{\omega^2}{\beta^2} - k_x^2\right)^{\frac{1}{2}}$$

The domain in which we will use the Green's operator is the (z,k_x,ω)-domain. Inverse transforming over k_z we have

$$G_o = \frac{1}{(2\pi)^{3/2}\rho_o} \int dk_z \; \tilde{\nabla} \begin{bmatrix} \frac{1}{\alpha^2(k_z-\nu)(k_z+\nu)} & 0 \\ 0 & \frac{1}{\beta^2(k_z-\eta)(k_z+\eta)} \end{bmatrix} \tilde{\nabla}^T \frac{e^{ik_x x'+ik_z(z'-z)}}{k_x^2+k_z^2} \tag{A4}$$

This integral can be easily evaluated by contour integration in the complex k_z-plane. For the exploding Green's operator we choose the pair of poles that makes $k_z(z'-z) < 0$. To satisfy the radiation condition, the contour is closed in the upper half-plane for $(z'-z) > 0$, and in the lower half-plane for $(z'-z) < 0$. Using the residue theorem we have

$$G_o = \frac{ie^{ik_x x'}}{(2\pi)^{\frac{1}{2}}\rho_o\omega^2}\left[\tilde{\nabla}_\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{\nabla}_\alpha^T \frac{e^{-i\nu|z'-z|}}{-2\nu} + \tilde{\nabla}_\beta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\nabla}_\beta^T \frac{e^{-i\eta|z'-z|}}{-2\eta}\right] \tag{A5}$$

where

$$\tilde{\nabla}_\alpha = i\begin{pmatrix} k_x & -\nu \\ \nu & k_x \end{pmatrix} \qquad \text{and} \qquad \tilde{\nabla}_\beta = i\begin{pmatrix} k_x & -\eta \\ \eta & k_x \end{pmatrix}$$

The first term in the Green's operator depends only on the compressional velocity (α), while the second depends only on the shear velocity (β). This leads to a natural definition for the two terms

$$\langle k_x,0|G_o|x',z'\rangle = \frac{ie^{ik_x x'}}{(2\pi)^{\frac{1}{2}}\rho_o\omega^2}\left[Ae^{-i\nu|z'|} + Be^{-i\eta|z'|}\right] \tag{A6}$$

where

$$A = \frac{-1}{2\nu}\tilde{\nabla}_{\alpha}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\tilde{\nabla}_{\alpha}^{T} \qquad \text{and} \qquad B = \frac{-1}{2\eta}\tilde{\nabla}_{\beta}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\tilde{\nabla}_{\beta}^{T}$$

The other Green's operator that we need is the one transformed over the input set of variables

$$\langle x',z'|G_0|k_x,0\rangle = \frac{ie^{-ik_x x'}}{(2\pi)^{\frac{1}{2}}\rho_0\omega^2}\left[Ae^{-i\nu|z'|} + Be^{-i\eta|z'|}\right] \tag{A7}$$

APPENDIX B: Fourier Transform Of The Scattering Potential

The scattering potential may be written as an operator in the form

$$V(x',x") = \left[c\omega^2 I + \nabla' \begin{pmatrix}\alpha^2 a & 0\\ 0 & \beta^2 b\end{pmatrix}\nabla'^T + 2\beta^2 H'\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}H'^T - 2\beta^2 H'^T\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}H'\right]\delta(x'-x") \tag{B1}$$

We now Fourier transform over x' and x", and integrate (trivially) over x":

$$V(k',k") = \frac{1}{(2\pi)^2}\int dx'\, e^{ik'\cdot x'}\left[c\omega^2 I + \nabla' \begin{pmatrix}\alpha^2 a & 0\\ 0 & \beta^2 b\end{pmatrix}\nabla'^T + 2\beta^2 H'\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}H'^T - 2\beta^2 H'^T\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}H'\right] e^{-ik"\cdot x'} \tag{B2}$$

We now integrate the second through fourth terms by parts to reverse the order of the leading operators and the exp(ik'·x'). This allows us to write down the Fourier transform by inspection:

$$V(k',k") = \frac{1}{(2\pi)^2}\left[c\omega^2 I - \nabla' \begin{pmatrix}\alpha^2 a & 0\\ 0 & \beta^2 b\end{pmatrix}\nabla"^T\right.$$

$$- 2\beta^2 H' \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} H''^T + 2\beta^2 H'^T \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} H'' \Bigg] \tag{B3}$$

where

$$a = a(\mathbf{k}'-\mathbf{k}''), \qquad b = b(\mathbf{k}'-\mathbf{k}''), \qquad \text{and} \qquad c = c(\mathbf{k}'-\mathbf{k}'').$$

The various terms in a, b, and c can be collected together to produce a final form for the scattering potential

$$V(k_x', k_z' | k_x'', k_z'') = \frac{\omega^2}{(2\pi)^2} \left[c\ I + a \frac{\alpha^2}{\omega^2} \begin{pmatrix} k_x' k_x'' & k_x' k_z'' \\ k_z' k_x'' & k_z' k_z'' \end{pmatrix} \right.$$

$$\left. + b \frac{\beta^2}{\omega^2} \begin{pmatrix} k_z' k_z'' & k_z' k_x'' - 2k_x' k_z'' \\ k_x' k_z'' - 2k_z' k_x'' & k_x' k_x'' \end{pmatrix} \right] \tag{B4}$$