ANISOTROPY DISPERSION AND WAVE MIGRATION ACCURACY

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There are two distinct types of errors in wave migration. Of greatest practical importance is ***frequency dispersion***, occurring when different frequencies propagate at different speeds. This may be reduced by improving the accuracy of finite difference approximations to differentials. Its ultimate cure is sufficient refinement of the differencing mesh. Of secondary importance, and the topic of the present study, is ***anisotropy dispersion***. This occurs when waves propagating in different directions do so at different speeds. It is remedied by the Muir square root expansion. As a practical matter, it is rarely necessary to go beyond the well-known 45-degree equation. It is not that we do not observe waves at steeper angles; we often do. The problem at large angles is that our knowledge of seismic velocity is seldom, if ever, sufficiently accurate to justify the trouble involved in using wide angle equations.

Anisotropy is commonly associated with propagation of light in crystals. In reflection seismology anisotropy is occasionally invoked to explain the discrepancy between borehole velocity measurements (vertical propagation) and velocity determined by normal moveout (horizontal propagation). It may also arise as an undesirable side effect in seismic calculation and data processing. This subject was analyzed in detail in SEP-8 and the results will be restated here.

Point Source Response

The ideal wavefront from a Huygen's secondary source is a semicircle. The secondary source that actually results from the 15-degree extrapolation equation is an ellipse. The secondary source that actually results from the 45-degree extrapolation equation is an interesting, heart-like shape. These are depicted in figure 1.

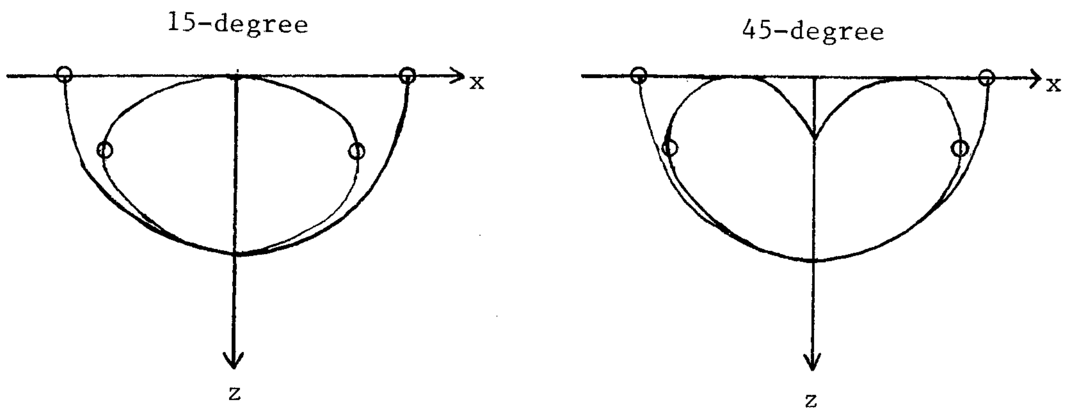

FIG. 1. Wavefronts of 15-degree and 45-degree extrapolation equations, inscribed within the exact semicircle. Waves with $\sin\theta = vk_x/\omega = \pm 1$ are marked with small circles.

In practice the top parts of the ellipse and the heart are rarely observed because they are in the evanescent zone, and the x-axis is seldom refined sufficiently for them to be below the aliasing frequency. The center of the heart is sometimes seen in the (x,t)-plane when the 45-degree program is used. It is depicted by a line drawing in figure 2 and shown by a 45-degree diffraction program in figure 3.

Wave Front Direction and Energy Velocity

In wave propagation we are familiar with the idea of energy propagating perpendicular to a wavefront. When there is anisotropy dispersion the two directions differ. The apparent horizontal velocity seen along the surface is dx/dt. The apparent velocity along the vertical, seen in a borehole, is dz/dt. Because of geometrical considerations, both of these apparent speeds exceed the wave speed. A vector perpendicular to the wavefront with a magnitude inverse to the velocity is called the slowness vector:

$$\text{slowness vector} \;=\; \left(\frac{dt}{dx}\,,\;\frac{dt}{dz}\right)$$

A vector perpendicular to the wavefront scaled to the speed of the wavefront is evidently the slowness vector divided by its squared magnitude. It is called the phase velocity vector:

$$\text{phase velocity} \;=\; \frac{\left(\frac{dt}{dx}\,,\;\frac{dt}{dz}\right)}{\left(\frac{dt}{dx}\right)^2 + \left(\frac{dt}{dz}\right)^2}$$

In a disturbance with sinusoidal form, the phase may be set equal to a constant, and the derivatives may be determined, giving

$$\text{slowness vector} \;=\; \left(\frac{k_x}{\omega}\,,\;\frac{k_z}{\omega}\right) \tag{1a}$$

The direction of energy propagation is somewhat more difficult to derive, but it turns out to be the so-called ***group velocity***.

$$\text{group velocity} \;=\; \left(\frac{\partial}{\partial k_x}\,,\;\frac{\partial}{\partial k_z}\right)\,\omega(k_x,k_z) \tag{1b}$$

For the scalar wave equation $\omega^2/v^2 = k_x^2 + k_z^2$, the group velocity and

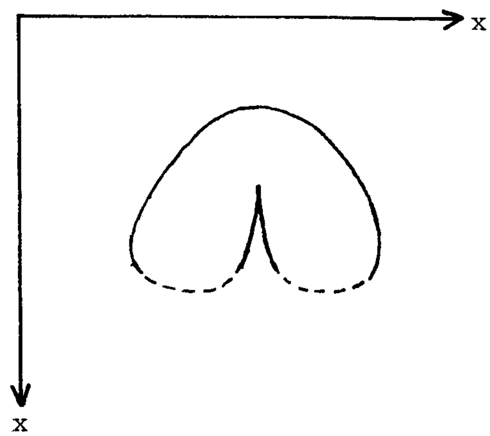

FIG. 2. 45-degree heart theory.

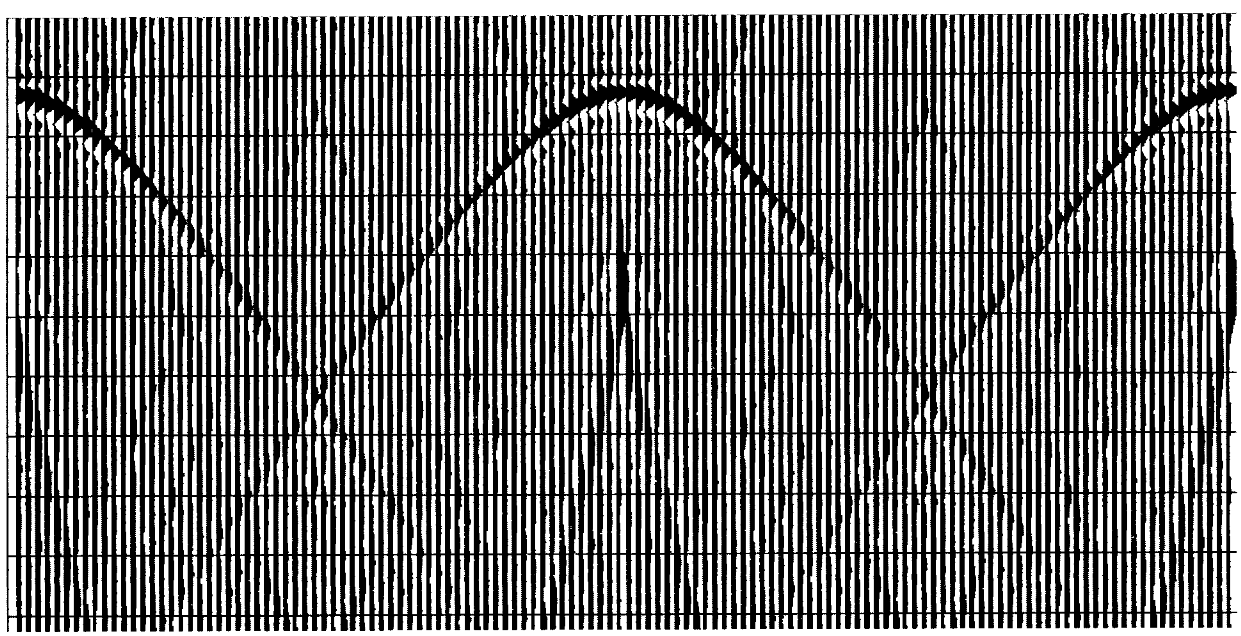

FIG. 3. 45-degree heart example.

the slowness vector are in the same direction, as may be verified with equation (1). The most familiar type of dispersion is frequency dispersion, where different frequencies travel at different speeds. It is shown in SEP-8 that the familiar (15-degree, 45-degree, etc.) extrapolation equations do not exhibit frequency dispersion. Specifically, as functions of ω and k_x/ω, the velocities do not depend on ω. In other words, the elliptical and heart shapes of figure 1 are not frequency-dependent.

The most interesting aspect of anisotropy dispersion is that energy appears to be going in one direction when it is actually going in another. To illustrate this phenomenon we will consider an exaggerated instance of it in which the group velocity has a downward component and the phase velocity has an upward component. Figure 4 depicts the dispersion relation of the 45-degree extrapolation equation. A slowness vector, which is a wavefront normal, has been selected by drawing an arrow from the origin to the dispersion curve. The corresponding direction of group velocity may now be determined graphically by noting that group velocity is defined by the gradient operator in equation (1b). Think of ω as the height of a hill where k_z points north and k_x points east. Then the dispersion relation is a contour of constant altitude. Different numerical values of frequency result from drawing figure 4 to different scales. The group velocity, in the direction of the gradient, is perpendicular to the contours of constant ω.

The anisotropy dispersion phenomenon is most clearly recognized in a movie, although it can be understood on a single frame, as in figure 5. Figure 6 is a line drawing interpreting energy flow from the top, through the prism, reflecting at the 45-degree angle, reflecting from the side of the frame, and finally entering an area of the figure which is sufficiently large and uncluttered for the phase fronts to be recognized as energy apparently propagating upward but actually propagating downward.

That neither energy nor information can propagate upward in figures 5 or 6 is obvious from the standpoint of the program which calculates figure 5. The program does not have the entire frame in memory; it

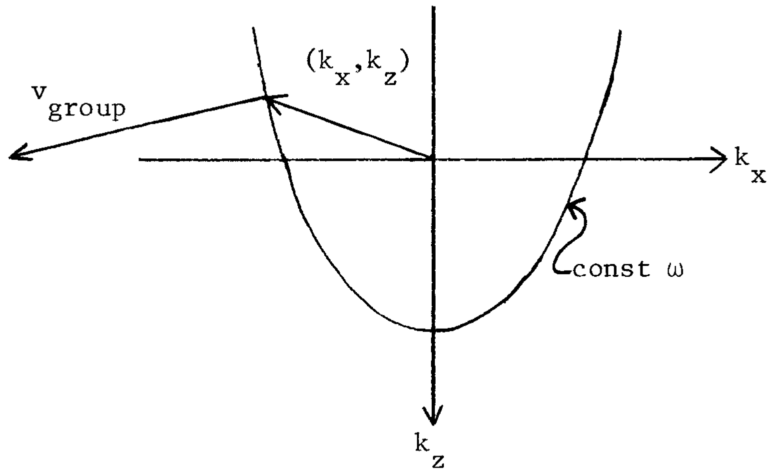

FIG. 4. Dispersion relation for downgoing extrapolation equation showing group velocity vector and slowness vector (perpendicular to wavefront).

produces one horizontal strip at a time from the strip just above. Thus the movie's phase fronts, which appear to be moving upward, seem very curious. Theoretically we do not expect wave extrapolation, particularly the 45-degree equation, to handle angles to 90 degrees. Yet the example shows that these extreme cases are indeed handled, although in a somewhat perverted way.

I once observed a similar circumstance on reflection seismic data from a geologically overthrusted area. The data could not be made available to me at the time, and by now it is probably long lost in the owner's files, so you will have to be content with the recollected line drawing in figure 7. The increasing velocity with depth causes the ray to bend upward and reflect from the underside of the overthrust. To see what is happening in the wave equation, it is helpful to draw the dispersion curve at two different velocities, as in figure 8. Downward continuation of a bit of energy with some particular stepout $dt/dx = k_x/\omega$ begins at a quite ordinary angle on the near surface, slow velocity dispersion curve. But as deeper velocity material is encountered at depth, that same stepout implies a negative phase velocity.

The situation resembles that in figures 5 and 6. Although the thrust angle is unlikely to be quantitatively correct, the general picture is appropriate.

Analyzing Errors of Migration

A dipping reflector that is flat and regular may be analyzed in its entirety with the phase velocity concept. The group velocity concept is required only when more than one angle is simultaneously present, as will be the case when we analyze the point scatterer response. In addition, a dipping bed with reflection amplitude variable along the bed must be analyzed with group velocity. Figure 9 depicts a smooth, flat, dipping bed which has been undermigrated because the $\hat{k}_z$ defined by some rational square root approximation or some numerical approximation did not match the correct square root value of k_z.

The error in this case is entirely a time shift error. Since in this case we have taken the reflection coefficient to be constant along the reflector, no lateral shift error can be recognized. The time error may be theoretically determined by

$$\frac{dt}{t} = \frac{\hat{k}_z - k_z}{k_z} \tag{2}$$

For the so-called 15-degree equation it turns out that about a half-percent phase error is made at 25 degrees.

Next, we determine the error in the collapse of a hyperbola. Figure 10 depicts the downward continuation of a hyperbola. For clarity the downward continuation was not taken all the way to the focus. We will keep track of a ray of some Snell's parameter $p = dt/dx$ by selecting some slope p and constructing a tangent line segment of slope p to each of the hyperboloids. If there were a little amplitude anomaly where the slope is p, you would be able to identify it on each of the hyperboloids. The group velocity is needed because either a curved event or an amplitude anomaly requires a range of plane wave angles to be represented. This is analogous to a time series wavelet's

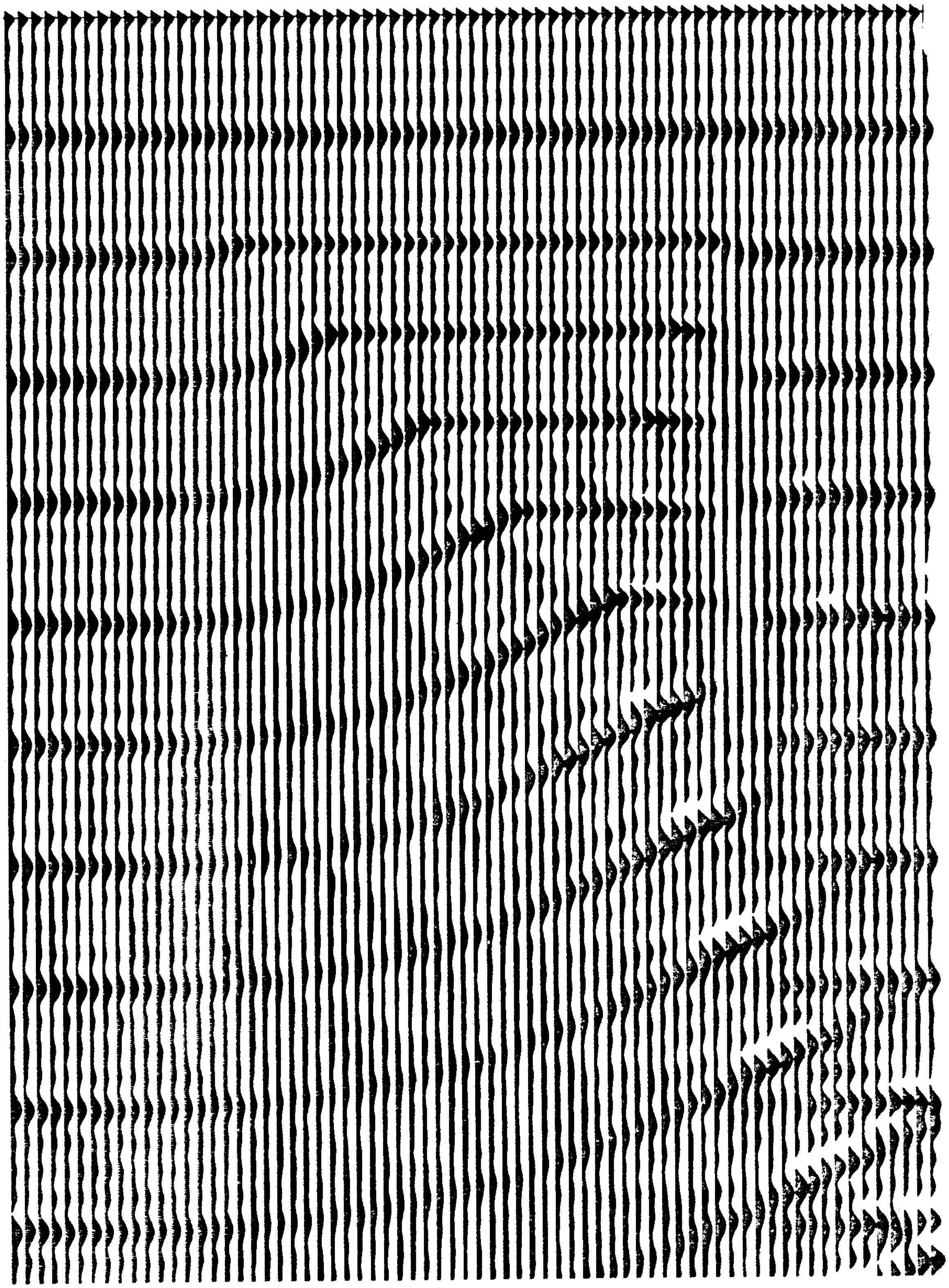

FIG. 5. Plane waves of four different frequencies propagating through a right, 45-degree prism. From SEP-1, p. 26: this is one frame from a movie made by Raul Estevez.

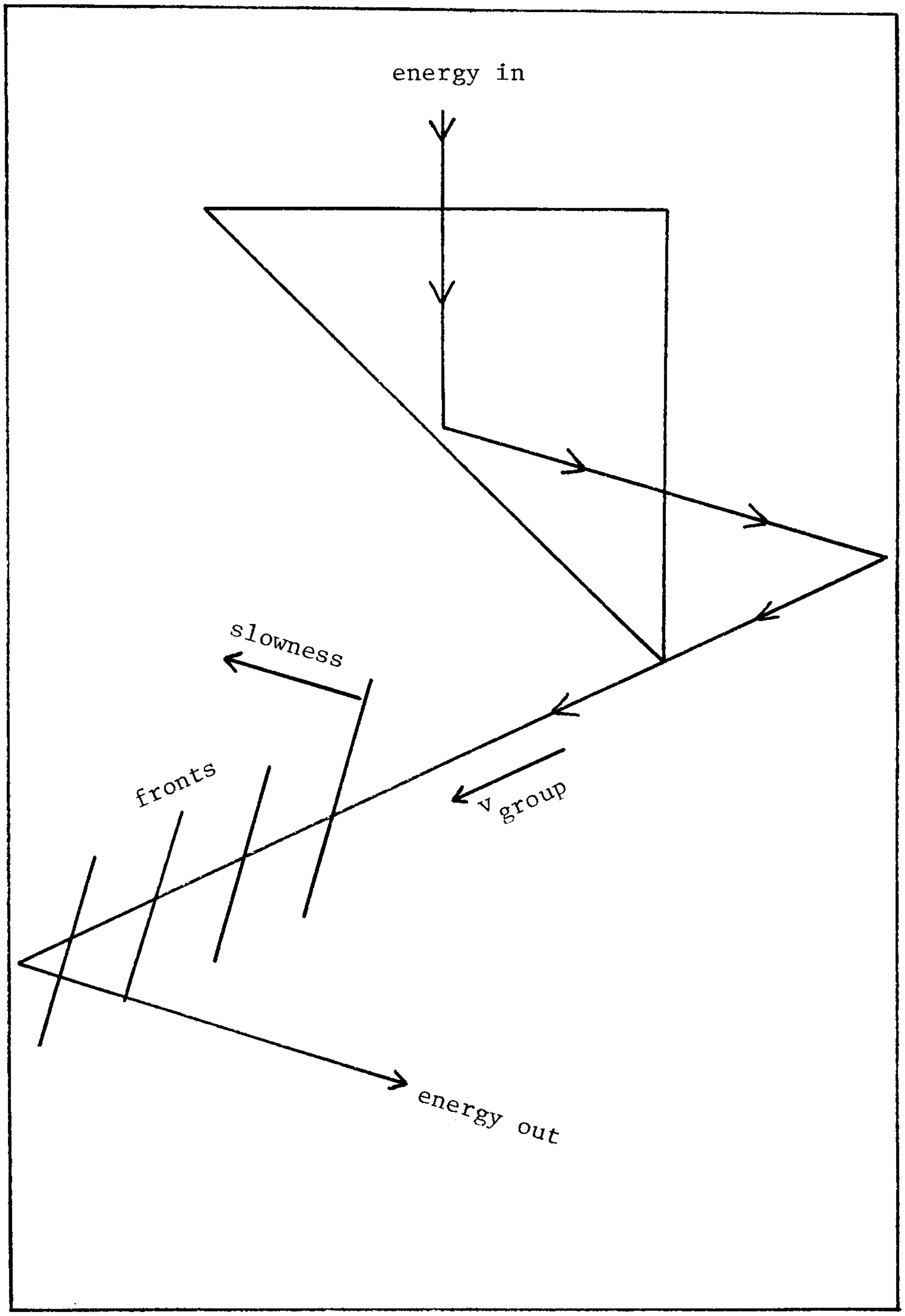

FIG. 6. Interpretation of some energy flow in figure 5 which illustrates different directions of energy and wavefront normal.

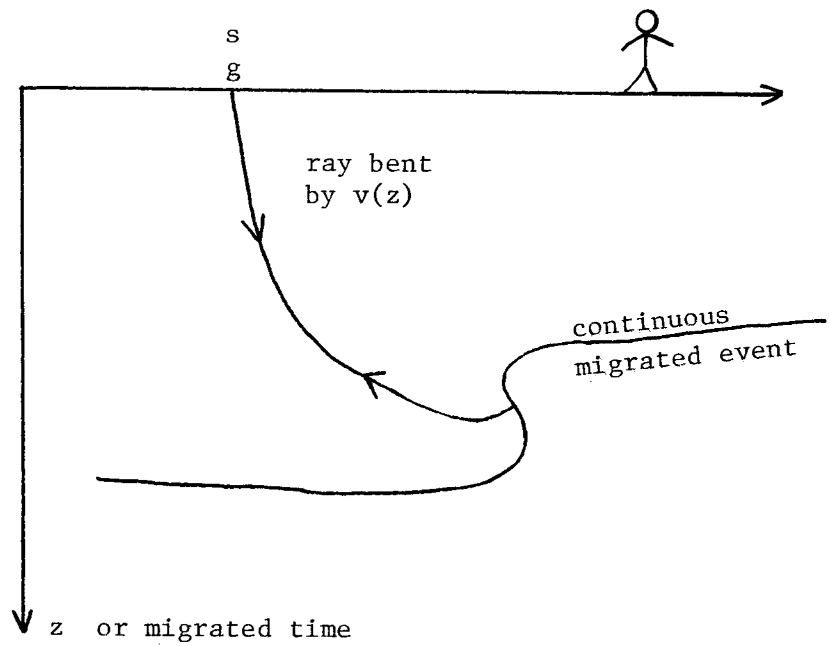

FIG. 7. Ray reflected from underside of overthrust.

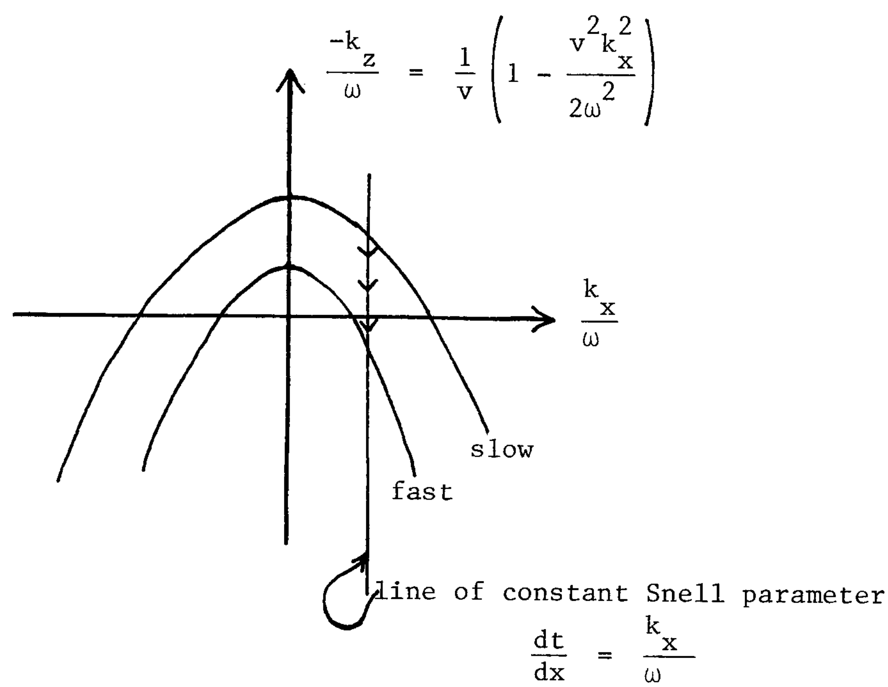

FIG. 8. Dispersion curve at two different velocities v_{fast} and v_{slow}.

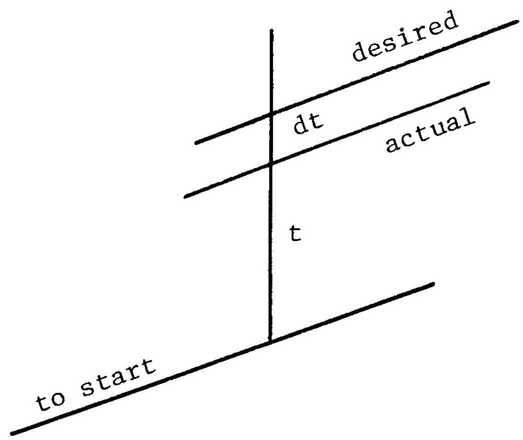

FIG. 9. Undermigrated dipping reflector.

need for a range of frequencies to represent it. You will notice in figure 9 that the actual amount of time moved is too little; likewise, the actual lateral distance moved is too small; so in practice, the errors are sometimes compensated for by about a six percent increase of either z or v. The actual amounts of the errors are shown to be

$$\frac{\Delta t}{t} = \frac{\frac{\partial}{\partial \omega}(\hat{k}_z - k_z)}{\frac{\partial}{\partial \omega} k_z} \tag{3a}$$

$$\frac{\Delta x}{x} = \frac{\frac{\partial}{\partial k_x}(\hat{k}_z - k_z)}{\frac{\partial}{\partial k_x} k_z} \tag{3b}$$

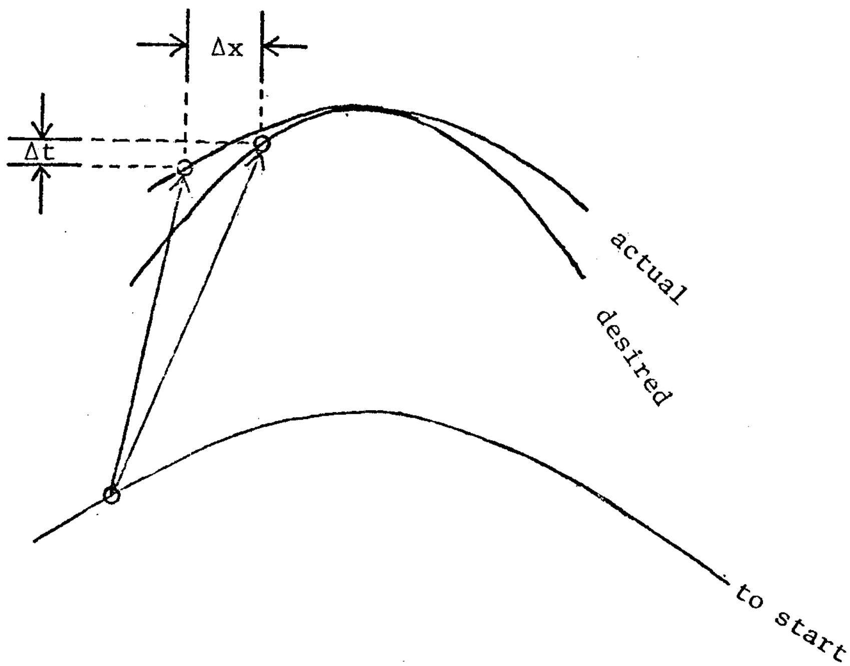

FIG. 10. Error of hyperbola collapse. Note that the actual curve is above the desired curve, but the actual point is below the desired point.

where k_z is taken to be a function of ω and k_x. It turns out that for the 15-degree equation, about a half-percent group velocity error occurs at 20 degrees. Thus the group velocity error is generally worse than the phase velocity error.

Derivation of Group Velocity Equation

We can make up an impulse function at the origin in (x,z)-space by superposing Fourier components:

$$\iint e^{+ik_x x + ik_z z} \, dk_x \, dk_z \tag{4}$$

Physics and possibly numerical analysis lead to a dispersion relation which is a functional relation between ω, k_x, and k_z, say, $\omega(k_x, k_z)$.

The most common example is the scalar wave equation $\omega^2 = (k_x^2 + k_z^2)/v^2$. The solution to the equations is

$$e^{-i\omega t + ik_x x + ik_z z} \tag{5}$$

Integrating (5) over (k_x, k_z) will produce a monochromatic time function which at $t = 0$ is an impulse at $(x,z) = (0,0)$. The expression at some very large time t is

$$\iint e^{-it\left[\omega(k_x,k_z) - k_x \frac{x}{t} - k_z \frac{z}{t}\right]} dk_x \, dk_z \tag{6}$$

At t-very large, the integrand is a very rapidly oscillating function of unit magnitude. Thus the integral will be nearly zero unless we can get the quantity in square brackets to become nearly independent of k_x and k_z for some sizable area in (k_x, k_z)-space. To find such a flat spot we proceed as if we were finding the max or min of a two-dimensional function, that is, by setting derivatives to zero. This analytical approach is known as the stationary phase method:

$$0 = \frac{\partial}{\partial k_x}[\;] = \frac{\partial\omega}{\partial k_x} - \frac{x}{t} \tag{7a}$$

$$0 = \frac{\partial}{\partial k_z}[\;] = \frac{\partial\omega}{\partial k_z} - \frac{z}{t} \tag{7b}$$

So in conclusion, at time t the disturbances will be located at

$$(x,z) = t\left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_z}\right) \tag{8}$$

which justifies the definition of group velocity.

Derivation of Energy Migration Equation

Energy migration in (x,t)-space is analyzed in a fashion similar to the derivation of group velocity. You take depth to be large in the integral

$$\iint e^{iz\left[k_z(\omega,k_x) - \omega \frac{t}{z} + k_x \frac{x}{z}\right]} \, d\omega \, dk_x \qquad (9)$$

The result is that the energy goes to

$$(x,t) \;=\; z\left(-\frac{\partial k_z}{\partial k_x}, \frac{\partial k_z}{\partial \omega}\right) \qquad (10)$$

This justifies our previous assertion that (3) can be used to analyze energy propagation errors.