THE CHOICE OF VARIABLES FOR ELASTIC WAVE EXTRAPOLATION

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Abstract

The choice of variables for elastic extrapolation problems determines the form of the one-way wave equations. Three sets of variables are considered: displacements, potentials, and a mixed set of variables which eliminate troublesome ∂_{xz} terms from the full wave equation. The displacements satisfy boundary conditions at internal interfaces but presently lack a recurrence relation to generate higher order approximation. The potential variables (P and S waves) appear to have problems similar to those of the displacements for variable velocity media, because the equation contains a complicated term which is difficult to factor into up- and downgoing waves. The mixed set of variables have the desired recurrence relation but fail to satisfy the boundary conditions. They also appear to be unstable for variable velocity.

Introduction

In this paper we discuss the choice of state variables for the modeling of elastic waves by extrapolation with one-way wave equations. One use of such extrapolation operators is in the finite difference migration of elastic wavefields. The main concern in migration is that the reflectors be placed in their correct positions, and usually the accuracy of the amplitudes is not of primary importance. However, with a little more care in the way velocity variations are incorporated into the extrapolation operators, one should be able to achieve accurate amplitudes as well as accurate traveltimes. This means that one may consider using extrapolation methods to provide solutions to forward modeling problems, such as refracted body waves and surface waves. In this SEP report, we have included a paper on modeling Love wave modes in laterally varying media by scalar wave equation extrapolation. Our goal is to extend the scalar methods to elastic wave problems. The first step is to settle on a set of state variables for the problem.

With the scalar wave equation, the question of the choice of state variables does not arise because the wave equation is already in its simplest form. For acoustic waves a pressure (or potential) variable is used, and for SH waves a displacement variable is used.

For elastic extrapolation there are several choices of variables. The first is the displacements themselves, which are governed by a coupled vector wave equation. For the constant velocity elastic wave equation, the most obvious choice of variables are the potential variables (P and S waves), which convert the coupled vector equations into a pair of uncoupled scalar problems. Another choice is a set of mixed variables ("mixed" for lack of a better name) that transform the equation into one that looks like a scalar wave equation with matrix coefficients. In the sections that follow, we will discuss the relative merits of each type of state variable.

To be useful an extrapolation operator has to have certain accuracy and stability properties. An *ideal* operator would have the following attributes:

1) The operator should have a recurrence relation which generates higher order approximations for increased angular accuracy. The existence of such an operator aids in the proof of stability of extrapolation, and gives one the confidence that, at least in principle, the method is not limited by accuracy considerations. In scalar theory this is accomplished by Muir's recurrence relation.

2) The operator should be stable for both constant and variable velocities. In elasticity it is not trivial for the operator to be stable for constant velocity because of the large differences between the shear and compressional velocities. Ideally one would like stability for arbitrary (and extreme) velocity variations. An example of an extreme variation in velocity that occurs frequently is the shear velocity when a water layer is included in the model. Stability appears to be linked with the following property.
3) The operator should implicitly match boundary conditions along internal interfaces. By implicit matching we mean that velocity gradients are included in the operators such that they mimic the behavior of the exact variable velocity wave equations at the interfaces. However, since we generally ignore transmission coefficients in the extrapolation direction, we cannot hope to exactly match boundary conditions for arbitrary interface orientations. As a minimum condition, the solution should be exact for layers parallel to the direction of extrapolation. The alternative is to match explicitly the boundary conditions by modifying the operator at the layer interfaces to preserve the continuity of stress and displacements. The explicit matching makes the operators more problem-specific, and the models more difficult to specify. In addition to the velocity function that the implicit matching requires, explicit matching requires the locations and orientations of the interfaces.

Displacement Variables

In elastic theory three types of variables are usually discussed: displacements, stresses, and potentials. If one also happens to be interested in differential operators, then displacements will be the fundamental set because both stresses and potentials are expressible as first order differentials of displacements. Thus, it is particularly easy to transform the displacement solution into stress to apply boundary conditions, or into potentials (P and S waves) for interpretation.

The problem with displacements is that the full operator (for constant density)

$$(\partial_z A\, \partial_z + \partial_z B\, \partial_x + \partial_x B^T \partial_z + \partial_x C\, \partial_x + \omega^2 I)\underline{u} = 0 \qquad (1)$$

$$\text{where} \quad \underline{u} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \text{horz. disp.} \\ \text{vert. disp.} \end{bmatrix}$$

$$A = \begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & \beta^2 \\ \alpha^2 - 2\beta^2 & 0 \end{bmatrix}, \; \text{and } C = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}$$

contains both first and second order differentials in z. Consequently, the formal factoring of the operator into one-way equations is

$$(\partial_z + D_1)(\partial_z - D_2)\underline{u} = 0. \qquad (2)$$

If the first order differentials were absent in equation (1), then D_1 would equal D_2, and life would be a lot simpler. Expanding equation (2) and matching it to a constant velocity form of equation (1) leads to the constraints

$$D_1 - D_2 = iA^{-1}(B + B^T)k_x$$

$$\text{and} \quad D_1 D_2 = A^{-1}(Ck_x^2 - I\omega^2)$$

Solving for D_1 and D_2 we have the quadratic equations

$$D_1^2 - ik_x D_1 A^{-1}(B + B^T) + A^{-1}(I\omega^2 - Ck_x^2) = 0 \qquad (3)$$

$$\text{and} \quad D_2^2 + ik_x A^{-1}(B + B^T)D_2 + A^{-1}(I\omega^2 - Ck_x^2) = 0. \qquad (4)$$

Thus, in order to build up a sequence of approximations for one-way displacement operators, the recurrence relation has to solve a quadratic rather than the simpler square root of scalar theory. We have not been successful in finding such a recurrence relation. The chief difficulty seems to be that the D matrices in equations (3) and (4) do not commute with either A, B, or C.

At present there exist both an exact solution and a second-order approximation to equations (3) and (4) (Clayton and Claerbout, SEP-15, p. 233-246). From scalar theory we know that higher order approximations obtained by Taylor series expansions of the exact solution will lead to unstable operators.

If a recurrence relation for displacement variables seems so difficult to obtain, the obvious question is why bother with them? The answer is that of all the variables considered in this paper, they come the closest to satisfying the internal boundary conditions. The boundary conditions for an elastic medium are continuity of stresses and displacements. For an interface parallel to the z-axis, these may be written as (for constant density)

$$\left[\begin{pmatrix} \alpha^2\partial_x & (\alpha^2-2\beta^2)\partial_z \\ \beta^2\partial_z & \beta^2\partial_x \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}\right] = 0 \quad \text{and} \quad \left[\begin{pmatrix} u \\ w \end{pmatrix}\right] = 0. \tag{5}$$

where the square brackets denote differences across the interface.

For a plane wave traveling in z-direction, the dominant term in the one-way approximation, as far as the boundary conditions are concerned, is $E\partial_{xx}$, where E is determined by matching the dispersion relation. If we write this as $E_1\partial_x E_2\partial_x$ the implicit boundary conditions are (see Brown, this report)

$$\left[\underline{u}\right] = 0 \quad \text{and} \quad \left[E_2\partial_x\underline{u}\right] = 0.$$

Thus if we choose $E_1 = E_2^{-1}E$ and

$$E_2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}$$

we will satisfy the correct boundary conditions [equation (5) with $\partial_z = 0$]. If the wave impinges at some other angle of incidence, then other terms in the one-way operator become significant. We have not yet analyzed this case.

In summary, displacement variables lack a recurrence relation to generate higher order approximations. They do, however, have the potential for matching the implicit boundary conditions.

Potential Variables

The displacement wave equation in constant velocity media can be diagonalized (decoupled) by introducing potentials of either the form

$$\underline{u} = \nabla P + \nabla \times S,$$

or the form

$$P' = \nabla \cdot \underline{u} \quad \text{and} \quad S' = \nabla \times \underline{u} \; .$$

The two are equivalent in the sense that

$$P' = -\frac{\omega^2}{\alpha^2} P \quad \text{and} \quad S' = -\frac{\omega^2}{\beta^2} S .$$

Consider writing equation (1) in the form

$$(H - J)\underline{u} = 0$$

where H is the homogeneous part (or constant velocity part) and J contains all the velocity variations. Conversion of this equation to P and S potentials amounts to diagonalizing the operator H. Defining the diagonalization as

$$Q^{-1} H \, Q = (\partial_{zz} - \Lambda^2) \quad \text{and} \quad \underline{p} = \begin{bmatrix} P \\ S \end{bmatrix} = Q^{-1} \underline{u},$$

the equation becomes

$$(\partial_{zz} - \Lambda^2 - Q^{-1}J\ Q)\underline{p} = 0.$$

We may formally factor the problem as

$$(\partial_z + \Lambda)(\partial_z - \Lambda)\underline{p} = Q^{-1}J\ Q\underline{p}. \qquad (6)$$

The right-hand side can be considered as a source term for the homogeneous operator on the left. The problem is sorting out what parts of the source term to retain. It is obvious, by the way the operator is factored, that the left-hand side contains waves moving in both directions in z. What is not so obvious is that the right-hand side also does. It contains reflection and transmission coefficients for both types of waves. It is important to eliminate the backscattering components from the source term. For example, if we inadvertently retain a reflection term for a backscattered wave, the extrapolation operator will start a new wave moving in the extrapolation direction.

The operator $Q^{-1}J\ Q$ is more complicated than the displacement equation itself. Thus, by transforming to P and S waves, it would appear that the problem has become more complicated.

Mixed Variables

In SEP-10 Claerbout and Clayton (p. 165 ff.) derive a 15-degree type equation for elastic waves which is cast in terms of the horizontal derivative of the vertical displacement w_x and the shear stress τ_{xz}. Wave equations written in terms of a "mixed" set of variables such as this turn out to have the advantage that corresponding one-way wave equations with arbitrarily accurate dispersion relations are particularly easy to find.

The equations of motion for a two-dimensional elastic medium in Cartesian coordinates can be written as five first-order partial differential equations in the variables $(1/\rho)\tau_{xx}$, $(1/\rho)\tau_{zz}$, $(1/\rho)\tau_{xz}$

(normalized stresses) and u,w (horizontal and vertical displacements). By eliminating the variable $(1/\rho)\tau_{xx}$ and differentiating two of the resulting equations by x, these equations can be written as a 4 by 4 system of equations in the following form:

$$\frac{\partial}{\partial z}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} \quad (7)$$

where $r = [\ \partial_x u\ ,\ (1/\rho)\tau_{zz}\ ]^T$ $s = [\ w\ ,\ (1/\rho)\partial_x\tau_{zx}\ ]^T$ and the matrix operators A and B are given by

$$A = \begin{pmatrix} -\partial_{xx} & \frac{1}{\beta^2} \\ \partial_{tt} & -1 \end{pmatrix},\quad B = \begin{pmatrix} \frac{2\beta^2-\alpha^2}{\alpha^2} & \frac{1}{\alpha^2} \\ \partial_{tt}-\frac{4\beta^2}{\alpha^2}(\alpha^2-\beta^2)\partial_{xx} & \frac{2\beta^2-\alpha^2}{\alpha^2}\partial_{xx} \end{pmatrix} \quad (8)$$

In the equation above, and in the following discussion we have assumed that the density ρ and the compressional and shear velocities α and β are constants. It is then simple to eliminate s from equation (7) to get a second-order partial differential equation in r. In a similar manner, a second-order partial differential equation in the variables $\tilde{s} = [\partial_x w\ ,\ (1/\rho)\tau_{xz}]^T$ can be derived. Another possibility is to begin again with the five first-order equations of motion and eliminate all variables except the normal stresses τ_{xx} and τ_{zz}. This results in another second-order partial differential equation describing elastic wave propagation. The property common to all of these second-order wave equations is their form. Each can be written in the form

$$\partial_{zz}q = (M_1\partial_{tt} + M_2\partial_{xx})q \quad (9)$$

The interesting thing about equation (9) is that it contains no cross-derivatives (∂_{xz} terms). Other than the fact that M_1 and M_2 are 2 by 2 matrices, and q is a vector with two elements, the form of (9) is identical to the scalar wave equation. In table 1, we give the matrices M_1 and M_2 and the state variables q for four differential equations of the form (9). In addition we include the matrix operator Q which can be used to diagonalize equation (9) (i.e. transform it to P and S variables). By making the change of variables

$$\tilde{q} \quad = \quad Q^{-1}q \tag{10}$$

equation (9) becomes a decoupled system. Note that both Q and Q^{-1} have an x-dependence of the form ∂_{xx} (or equivalently k_x^2). This means that both the forward and inverse transformations can be implemented with a tri-diagonal operator in the (x,ω)-domain.

The fact that equation (9) looks like the scalar wave equation is what makes the derivation of the corresponding one-way wave equations so simple. For convenience, let us Fourier transform (9) over t and x to get

$$(\partial_{zz} + M_1\omega^2 + M_2k_x^2)q \quad = \quad 0. \tag{11}$$

Letting $M^2 = -(M_1\omega^2 + M_2k_x^2)$, equation (11) becomes

$$(\partial_{zz} - M^2)q \quad = \quad 0. \tag{12}$$

Since ρ, α, and β are assumed constant, M (the square root of M^2) will commute with ∂_z, and consequently (12) can be written in the form

$$(\partial_z + M)(\partial_z - M)q \quad = \quad 0. \tag{13}$$

Equation (13) shows that the factorization of the elastic wave equation operator into two one-way operators is straightforward for equations of the form (9). The approach we can take to find one-way equations corresponding to (9) is similar to the scalar case.

For the scalar (acoustic) equations, approximations to square-roots of differential operators are given by Muir's recursion formula, which can be written as

$\partial_{zz} q = (M_1 \partial_{tt} + M_2 \partial_{xx}) q$			
q	M_1	M_2	Q
$\begin{pmatrix} \partial_x w \\ \frac{1}{\rho}\tau_{xz} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\alpha^2} & 0 \\ -2\frac{\alpha^2-\beta^2}{\alpha^2} & \frac{1}{\beta^2} \end{pmatrix}$	$\begin{pmatrix} \frac{\alpha^2-2\beta^2}{\alpha^2} & -\frac{\alpha^2-\beta^2}{\alpha^2\beta^2} \\ 4\frac{\beta^2}{\alpha^2}(\alpha^2-\beta^2) & -\frac{3\alpha^2-2\beta^2}{\alpha^2} \end{pmatrix}$	$\begin{pmatrix} 1 & k_x^2 \\ 2\beta^2 & -\omega^2+2\beta^2 k_x^2 \end{pmatrix}$
$\begin{pmatrix} \frac{1}{\rho}\tau_{zz} \\ \partial_x u \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\alpha^2} & -2\frac{\alpha^2-\beta^2}{\alpha^2} \\ 0 & \frac{1}{\beta^2} \end{pmatrix}$	$\begin{pmatrix} \frac{\alpha^2-2\beta^2}{\alpha^2} & 4\frac{\beta^2}{\alpha^2}(\alpha^2-\beta^2) \\ -\frac{\alpha^2-\beta^2}{\alpha^2\beta^2} & -\frac{3\alpha^2-2\beta^2}{\alpha^2} \end{pmatrix}$	$\begin{pmatrix} k_x^2 & 1 \\ -\omega^2+2\beta^2 k_x^2 & 2\beta^2 \end{pmatrix}$
$\begin{pmatrix} \tau_{xx} \\ \tau_{zz} \end{pmatrix}$	$\begin{pmatrix} \frac{\alpha^2+2\beta^2}{2\alpha^2\beta^2} & -\frac{\alpha^2-2\beta^2}{2\alpha^2\beta^2} \\ -\frac{\alpha^2}{2\alpha^2\beta^2} & \frac{\alpha^2}{2\alpha^2\beta^2} \end{pmatrix}$	$\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \omega\frac{2\alpha^2-2\beta^2}{2\alpha^2\beta^2}+k_x^2 & 1 \\ \frac{\omega^2}{2\beta^2}-k_x^2 & -1 \end{pmatrix}$
$\begin{pmatrix} \tau_{xx}+\tau_{zz} \\ \tau_{xx}-\tau_{zz} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\alpha^2} & 0 \\ \frac{1}{\alpha^2} & \frac{1}{\beta^2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$	$\begin{pmatrix} \omega\frac{2\alpha^2-\beta^2}{\alpha^2\beta^2} & 0 \\ -\frac{\omega^2}{\alpha^2}+2k_x^2 & -\omega^2+2\beta^2 k_x^2 \end{pmatrix}$

TABLE 1. The coefficient matrices for four of the mixed variables are given in the table. In the first column are the state variables. The second and third columns are the coefficient matrices of the wave equation given at the top. The last column is the transformation operator which converts P and S potentials into the given variable.

$$S_{n+1} = \frac{\frac{i\omega}{v} S_n + (k_x^2 - \frac{\omega^2}{v^2})}{\frac{i\omega}{v} + S_n} \tag{14}$$

where S_n is the operator on the right-hand side of the scalar one-way wave equation:

$$\frac{\partial p}{\partial z} = S_n P. \tag{15}$$

In an analogous fashion, we can try the following recursion for the elastic case:

$$S_{n+1} = (L_n + S_n)^{-1}(L_n S_n + M^2) \tag{16}$$

Each of the elements in the recursion is now a 2 by 2 matrix operator. L_n is a matrix which can be chosen arbitrarily, because, as we will demonstrate below, if (16) converges to the exact square root, it will do so independently of the choice of L_n. Assume that (16) does indeed converge, and take its limit to be S_∞, i.e. $S_n \longrightarrow S_\infty$. Then

$$S_\infty = (L_\infty + S_\infty)^{-1}(L_\infty S_\infty + M^2)$$

from which it follows that

$$S_\infty^2 = M^2,$$

hence, if the recursion converges, it converges to the square root of M^2.

Equation (16) is a useful generalization of the Muir recursion even for the scalar case. This is because the matrix L_n may be used to improve the fit of the dispersion curve to the exact semi-circles of the "exact square-root" equations. To demonstrate this, first diagonalize (16). We will make the assumption that all of the matrices in equation (16) have the same set of eigenvectors. This is a reasonable assumption to make, since ideally we would like all the approximations to have the same transformation to diagonal form. It is also fairly easy to satisfy if we restrict all of the L_n to have the same eigenvectors as M. Then we only need to require that S_0 have those eigenvectors, and all S_n will have this property. Once we have transformed (16) to diagonal form, we can look at each equation separately:

$$s_{n+1} = \frac{\lambda_n s_n + m^2}{\lambda_n + s_n}, \qquad (17)$$

where λ_n is an eigenvalue of L, s_n is an eigenvalue of S_n and m is an eigenvalue of M. Note that the dispersion relation corresponding to (17) can be written as

$$ik_z^{(n+1)} = \frac{\lambda_n s_n + m^2}{\lambda_n + s_n}. \qquad (18)$$

Also note that m^2 has the form

$$m^2 = k_x^2 - \frac{\omega^2}{v^2}$$

and hence $m^2 = -(k_z^{(\infty)})^2$. Now if, at any point on the dispersion curve, $\lambda_n = ik_z^{(\infty)}$, then substituting into (18) we get that

$$ik_z^{(n+1)} = \frac{ik_z^{(\infty)}(s_n + ik_z^{(\infty)})}{ik_z^{(\infty)} + s_n} = ik_z^{(\infty)}.$$

In other words, the dispersion curve for S_{n+1} will fit the exact dispersion curve precisely at that point. Also, if at any point on the dispersion curve it should happen that $s_n = ik_z^{(\infty)}$, then the same thing will happen. We can use these properties to build up a sequence of approximations that precisely fit the exact dispersion relation at many points. If, for n = 0, we pick λ_0 so that there is an "exact fitting point" somewhere on the dispersion curve, then s_1 will have that exact fitting point as well, and so will s_2. Hence, all of the s_n will have that fitting point by the argument above. This means that when constructing s_2 we can choose λ_1 in order to fit the exact dispersion curve at some other point. So if s_0 fits exactly at one point, the n-th dispersion curve can have as many as n+1 exact fitting points.

Figure 1 shows dispersion curves for the equation corresponding to the first three approximations of the one-way elastic operator for some simple choices of S_0 and L_n in (16). In each case we have chosen all the L_n to be the same, and both S_0 and L_n to be scalars times the identity matrix. In figure 1a, $L = L_0 = L_1 = L_2 = \alpha^{-1}I$ and $S_0 = \beta^{-1}I$. Hence we expect $k_z = \alpha^{-1}$ and $k_z = \beta^{-1}$ to be exact fitting points for both the P and S dispersion curves, as they are. In figure 1b, $L = \beta^{-1}I$ and $S_0 = \alpha^{-1}I$. Thus, the exact fitting points are in the same place. Note, however, that the dispersion curves in a and b are not quite the same. In figure 1c, $S_0 = \alpha^{-1}I$ and $L = \frac{1}{2}\alpha^{-1}I$. The exact fitting points on both curves are therefore at $k_z = \alpha^{-1}$ and $k_z = \frac{1}{2}\alpha^{-1}$. As a result, the S-wave curve does not fit at all at the top of the semi-circle, so vertically traveling plane S-waves governed by this equation would have an incorrect phase velocity. In figure 1d we see that it is possible to obtain a range of near-tangency of the approximate curve to the exact dispersion curve by choosing two nearby fitting points. For this curve $S_0 = \alpha^{-1}$ and $L = 1.15\ \alpha^{-1}I$.

If in equation (17) above we choose $\lambda_0 = \lambda_1 = \cdots = -i\omega/v$, we recover Muir's usual recursion formula; and hence, the differential equations associated with the rational approximations to the square root will have dispersion curves which are the same as those of the corresponding scalar equations. The one difference is that the form of the one-way equations will be slightly different. The "5-degree" equation corresponding to the variables $s = [\partial_x u, (1/\rho)\tau_{xz}]^T$ will be

$$\frac{\partial s}{\partial z} = Q\begin{pmatrix} \frac{-i\omega}{\alpha} & 0 \\ 0 & \frac{-i\omega}{\beta} \end{pmatrix}Q^{-1}s \qquad (19)$$

which when the matrices Q are substituted in becomes

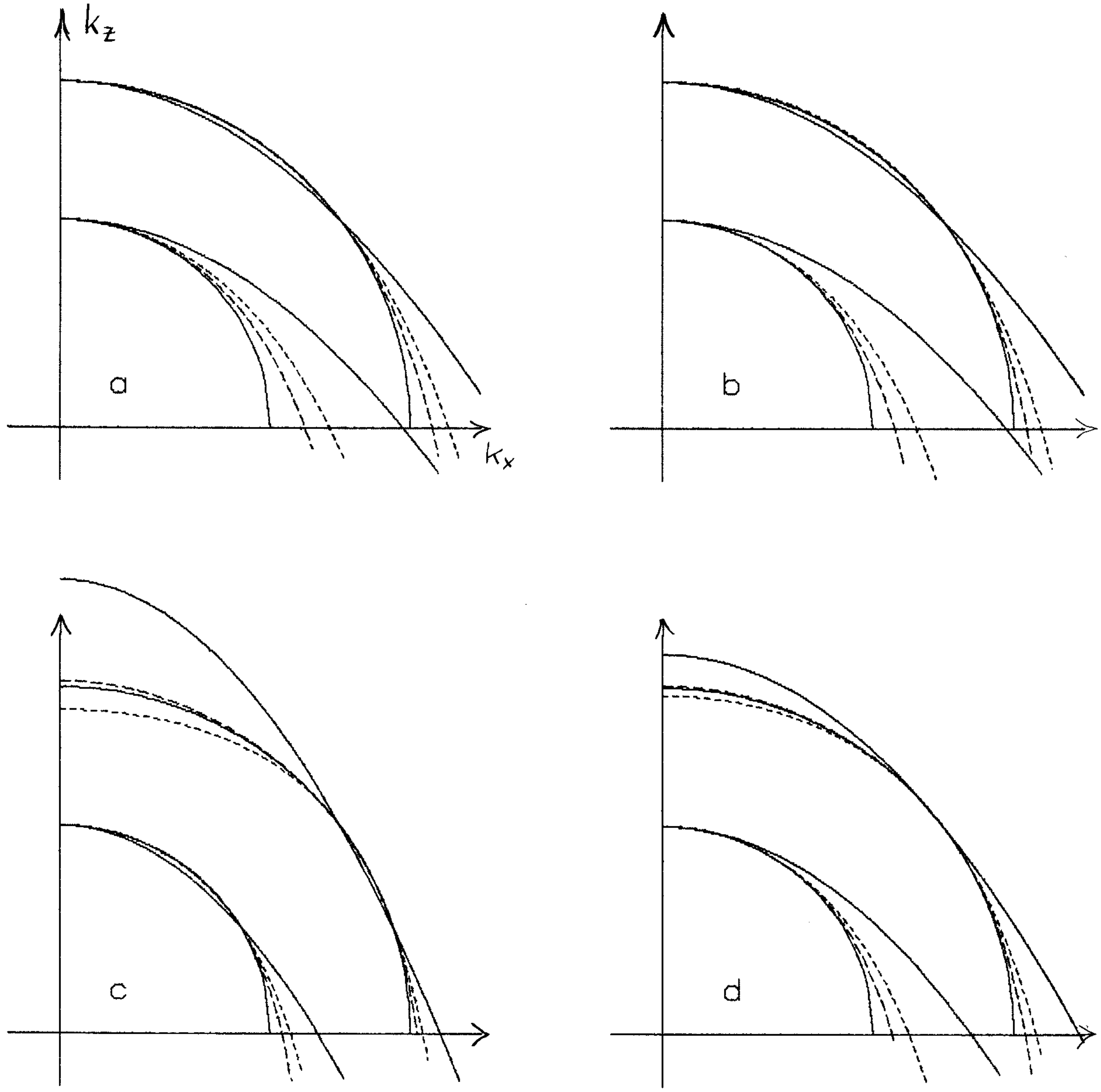

FIG. 1. Dispersion curves for various choices of S_o and L in the recursion formula

$$S_{n+1} = (L + S_n)^{-1}(LS_n + M^2)$$

for the square-root operator in the variables $q = [\partial_x u, (1/\rho)\tau_{zz}]^T$.

In a) $S_o = \beta^{-1}I$, $L = \alpha^{-1}I$; in b) $S_o = \alpha^{-1}I$, $L = \beta^{-1}I$;

in c) $S_o = \alpha^{-1}I$, $L = \frac{1}{2}\alpha^{-1}I$; in d) $S_o = \alpha^{-1}I$, $L = 1.15\alpha^{-1}I$. The curves are: S_1 (solid line), S_2 (short-dash line), and S_3 (long-dash line).

$$\frac{\partial s}{\partial z} = i\left\{\begin{bmatrix} -\frac{1}{\alpha} & 0 \\ \frac{2\beta}{\alpha}(\beta-\alpha) & -\frac{1}{\beta} \end{bmatrix}\omega + \begin{bmatrix} \frac{2\beta}{\alpha}(\alpha-\beta) & (\frac{1}{\alpha}-\frac{1}{\beta}) \\ \frac{4\beta^2}{\alpha}(\beta-\alpha) & \frac{2\beta}{\alpha}(\beta-\alpha) \end{bmatrix}\frac{k_x^2}{\omega}\right\} s. \qquad (20)$$

Note that (20), when inverse Fourier-transformed over x, will contain a double x-derivative. The form of (20) is that of the 15-degree scalar equation while the dispersion relation is only as good as the 5-degree equation. Thus, the equations derived from these recursion formulae can be more expensive computationally than their scalar counterparts.

There is another way to derive approximations to the "exact square root" elastic equation which also has the desirable feature that it gives one-way equations which have the same form *and* dispersion relations as their scalar counterparts. This method is simply to assume a form for the differential equation and then to match coefficients with a Taylor series expansion of the exact square root equation.

If the full elastic wave equation (in some set of variables) can be written as

$$(\partial_{zz} - Q\Lambda_m^2 Q^{-1})q = 0, \qquad (21)$$

then by the arguments following equation (12) above, the corresponding "exact" one-way equation can be written as

$$(\partial_z - Q\Lambda_m Q^{-1})q = 0 \qquad (22)$$

(where Λ_m is a diagonal matrix). We can now look for an approximate one-way equation by first assuming a form for that equation and then matching term for term with a Taylor series expansion of (22). For instance, let us look for an equation which has the same form as the scalar 45-degree equation:

$$(I + B_2 k_x^2)\partial_z q = (B_o + B_1 k_x^2)q \qquad (23)$$

We first expand equation (22) in a Taylor series about $k_x^2 = 0$:

$$\partial_z q = \sum_{n=0}^{\infty} D_n k_x^{2n} q \qquad (24)$$

We then match equations (23) and (24) by multiplying (24) by $(I + B_2 k_x^2)$ and then equating the left-hand sides of the two equations:

$$(B_o + B_1 k_x^2) = (I + B_2 k_x^2)(D_o + D_1 k_x^2 + D_2 k_x^4) + O(k_x^6) \qquad (25)$$

To find B_o, B_1 and B_2 we get three matrix equations to solve:

$$B_o = D_o \ , \quad B_2 D_o + D_1 - B_1 = 0 \ , \quad \text{and} \quad B_2 D_1 + D_2 = 0.$$

from which it follows that

$$B_2 = -D_2 D_1^{-1} \ , \quad B_1 = B_2 D_o + D_1 \ , \quad \text{and} \quad B_o = D_o. \qquad (26)$$

The next higher order equation in the scalar recursion takes the form

$$(I + C_2 k_x^2)\partial_z q = (C_o + C_1 k_x^2 + C_2 k_x^4) q. \qquad (27)$$

The coefficient matrices in (27) can be determined in the same way. We have done this numerically for equations (23) and (27) and also for the one-way elastic 15-degree-type equation derived by Claerbout and Clayton in SEP-10. The state variables we used were $[\partial_x u \ , (1/\rho)\tau_{zz}]^T$. We calculated their dispersion relation; they are plotted in figure 2. By graphical comparison, we determined that the two dispersion curves for each of the equations have exactly the same shape as the dispersion curve for the corresponding scalar equation.

The one-way elastic wave equations that we have derived in this section can be useful to us only if they are stable. Moreover, if we are interested in accurate modeling of elastic wave propagation, we would like the equations to be "bullet-proof" as well (stable for problems with strong lateral velocity variation) and to mimic the proper reflection and refraction effects for elastic waves at internal

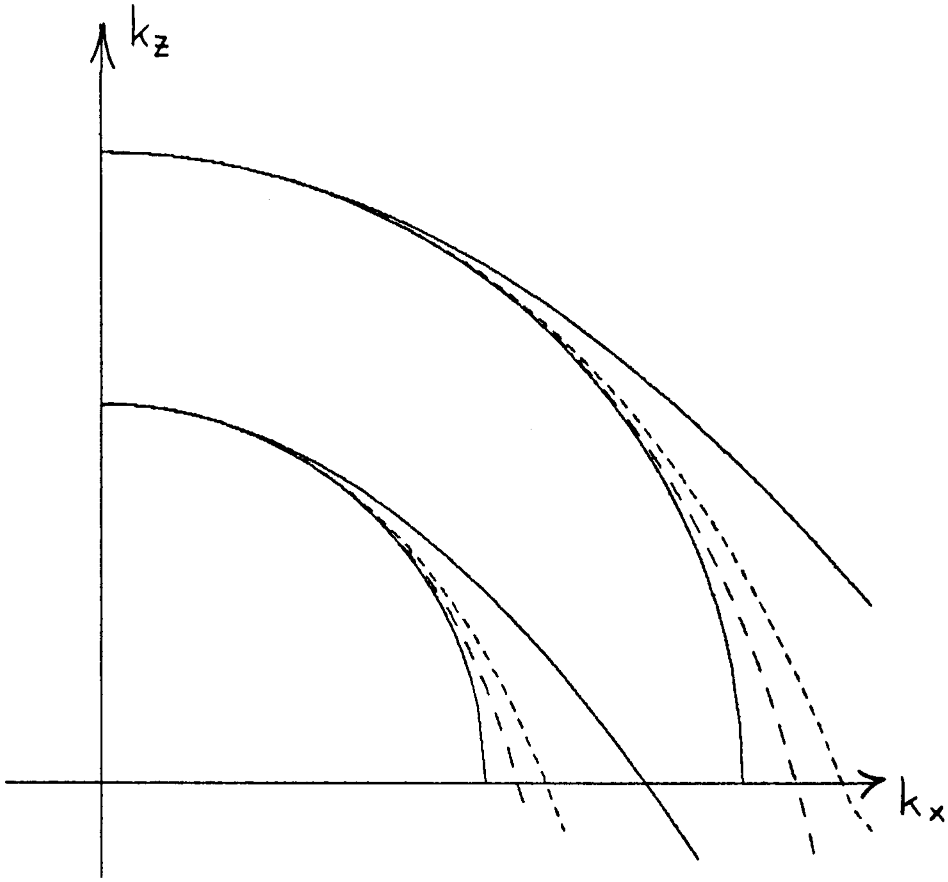

FIG. 2. Dispersion curves for one-way elastic wave equations in the variables $[\partial_x u, (1/\rho)\tau_{zz}]^T$ derived by matching coefficients. The curves are identical to the dispersion curves for the scalar counterparts of these equations (which are given by Muir's recursion formula). The solid line is the first order approximation; the short-dash line is the second order; and the long dash line is the third order.

interfaces. We have done some numerical experiments that show that we can find one-way equations in any of the variables mentioned in the table above that are stable for the constant coefficient case. Figure 3 shows the results of one such calculation. To produce these plots we solved an equation in the variables $q = [\partial_x u, (1/\rho)\tau_{zz}]^T$ of the form

$$\partial_z q = (L + S_0)^{-1}(LS_0 + M^2)q \tag{28}$$

where

$$L = \begin{pmatrix} i\frac{\omega}{\alpha} & 0 \\ 0 & i\frac{\omega}{\beta} \end{pmatrix}, \quad S_0 = \begin{pmatrix} i\frac{\omega}{\alpha} & 0 \\ 0 & i\frac{\omega}{\beta} \end{pmatrix}$$

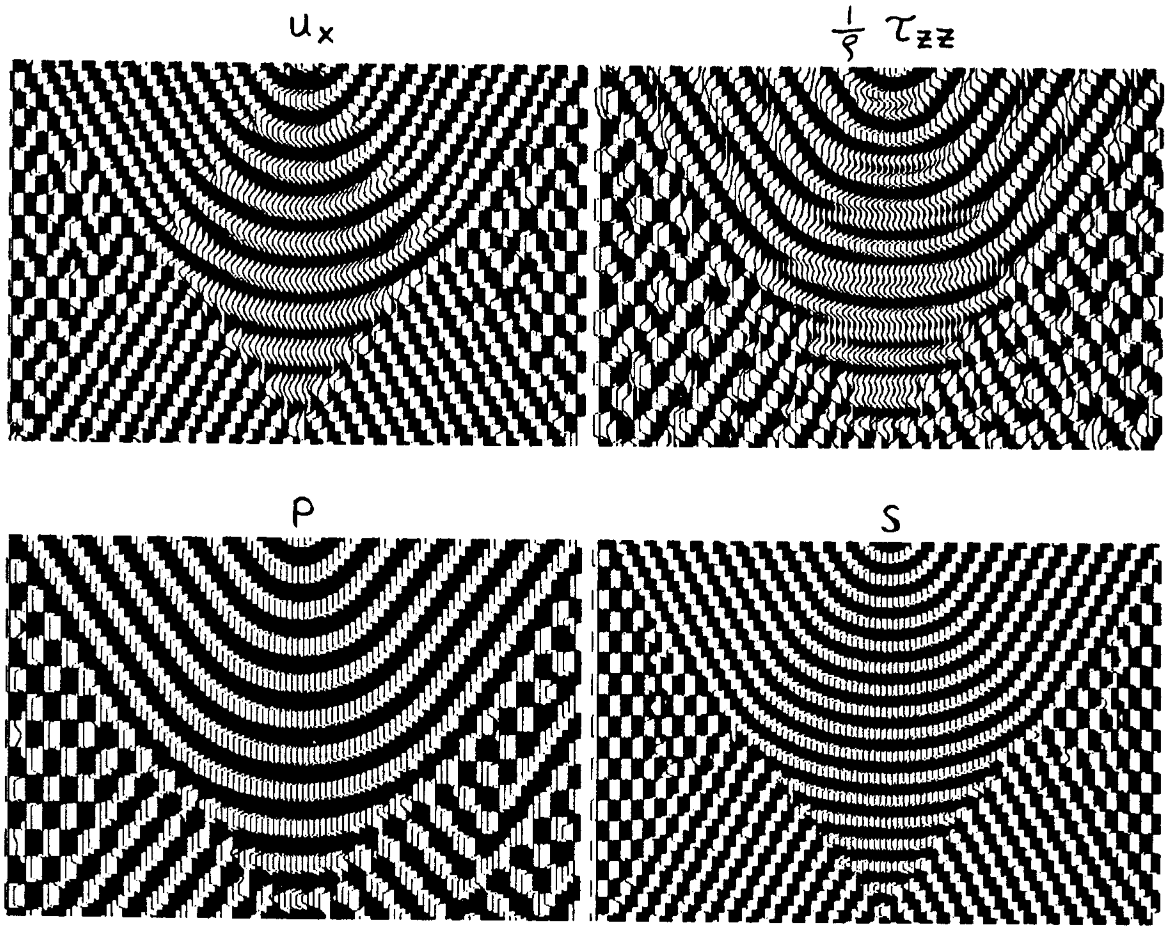

FIG. 3. An example of extrapolation with the first set of mixed variables listed in table 1. The initial conditions were constructed from a point source that is equal strength in both P and S. The Q transformation given in table 1 was then applied to provide initial conditions for the mixed variables. The top two panels show the extrapolation using the mixed variables. The bottom two panels show the results of transforming the solution to P and S. The P and S results are virtually identical with those obtained by scalar extrapolation.

and M^2 is given in the table [see equations (12) and (13)]. The initial conditions at $z = 0$ corresponded to the analytic solution of the full elastic wave equation for a point source of equal strength in P and S placed above the top of the plot. We used a Crank-Nicolson approximation to (28) to solve this problem. The top two plots show the solution in the variables $\partial_x u$ and $(1/\rho)\tau_{zz}$. The bottom two plots show the solution transformed to P and S. The main thing to note is that the solution is stable. Figure 4 shows similar plots, but this time a vertical velocity interface was placed in the medium slightly to the right of the point source. We checked the discrete L_2-norm of the solution at each z-step and found that it grew exponentially fast, indicating that the equation is unstable. This is apparent in the plots.

We now believe that all such equations are probably unstable for the case of strong lateral velocity variation. This belief is based on an approach to stability discussed by Brown in the section on internal boundary conditions in "Muir's Rules for Matrices ... " (this report). In that paper it is pointed out that the finding of bullet-proof approximations to differential equations is closely related to the idea of finding difference approximations that produce reasonable reflection and transmission effects at internal interfaces. If the reflection and transmission coefficients are of reasonable size, then the solution will not blow up when energy strikes the interfaces, i.e. the method is stable. As long as we are going to require that the method yield reasonable reflection and transmission coefficients, we might want to require that these coefficients be good approximations to the reflection and transmission coefficients for the full elastic problem. This seems to be where the biggest problem is with all the variable sets given in the table above. It appears that it is not possible to write down conditions that are equivalent to the requirements that all stresses be continuous at an interface using only the two variables in each set. According to the arguments in the paper mentioned above, if we can write down the interface conditions explicitly for a vertical interface, then we can find a difference approximation that automatically satisfies these conditions. Since we cannot write down those conditions explicitly, the outlook for being able to write down a physically reasonable bullet-proof approximation seems now to be quite bleak.

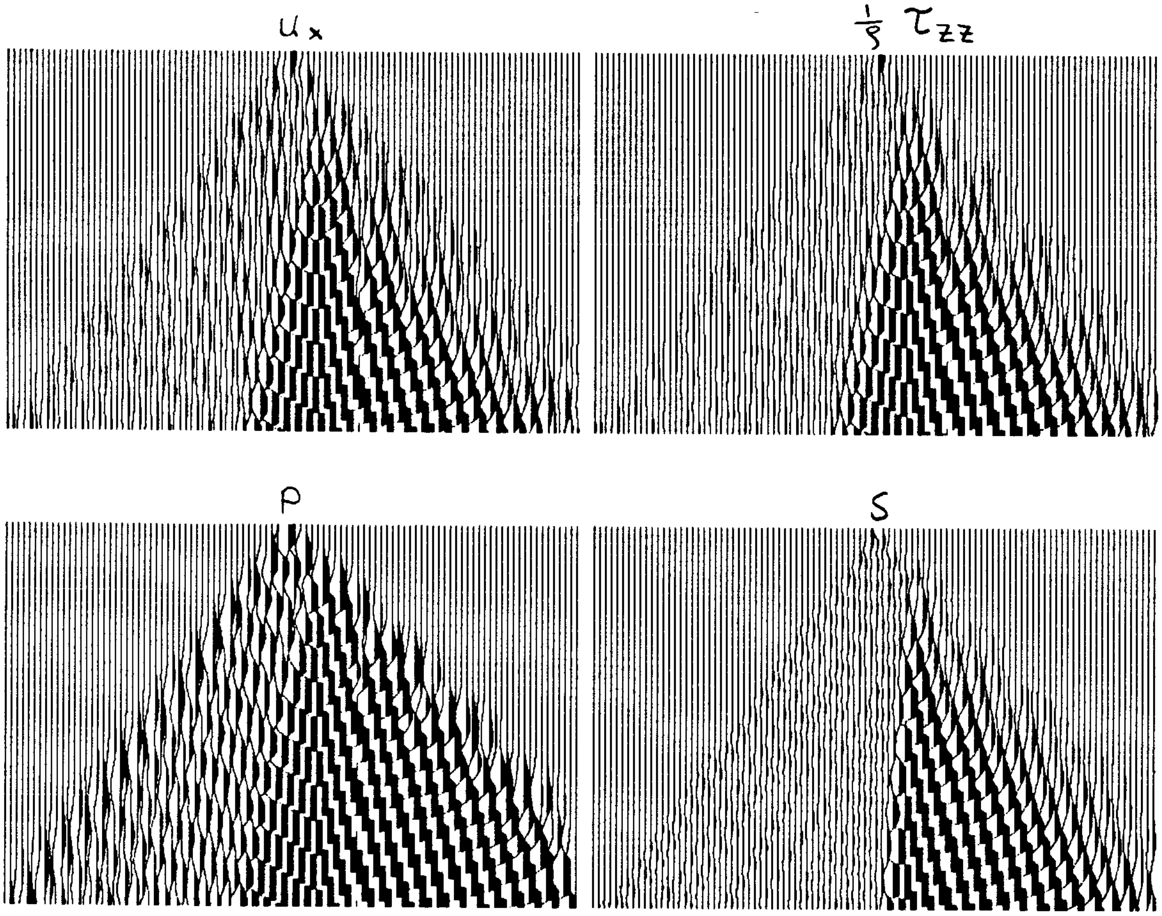

FIG. 4. This figure shows an example of an instability in the the mixed variable extrapolation. The top panel show the extrapolation using the mixed variables listed first in table 1. A vertical interface with a 30% velocity contrast is located just to the right of the point source. The waves transmitted and reflected by the interface are growing exponentially. The bottom panel shows the solution transformed to P and S waves.

The difference approximation used to produce the plots in figure 4 was one which implicitly specified the conditions that q and $\partial_x q$ be continuous across the vertical velocity interface. (An explanation for why these conditions are implicitly satisfied can be found in the paper by Brown mentioned above.) It is fairly simple to calculate explicitly what the reflection and transmission coefficients are for these interface conditions. It is conventional to calculate the ratio of the reflected and refracted P and S wave amplitudes to the incident P or S wave amplitude. So we need first to transform from the variables q to the potential variables P and S. Finding the potential variables amounts to diagonalizing the wave equation (12). Define a diagonal matrix Λ_m by

$$\Lambda_m = Q^{-1}MQ \tag{29}$$

Then if we define new variables $\tilde{q}$ by

$$\tilde{q} = Q^{-1}q, \tag{30}$$

equation (12) becomes

$$\frac{\partial^2\tilde{q}}{\partial z^2} - \Lambda_m\tilde{q} = 0. \tag{31}$$

We will only calculate the reflection and transmission coefficients for an incident P-wave. A plane incident P-wave has the form

$$\tilde{q}_o = [A \exp i(k_z z + k_{\alpha_1} x), 0]^T,$$

where k_{α_1} is the horizontal wave-number of the incident P-wave in the first medium. The reflected wave will have the form

$$\tilde{q}_R = [R_p \exp i(k_z z - k_{\alpha_1} x), R_s \exp i(k_z z - k_{\beta_1} x)]^T,$$

and the refracted wave will have the form

$$\tilde{q}_T = [T_p \exp i(k_z z + k_{\alpha_2} x), T_s \exp i(k_z z + k_{\beta_2} x)]^T .$$

Here k_{α_2} is the horizontal wavenumber of a P-wave in the second medium, and k_{β_1} and k_{β_2} are the horizontal wavenumbers of an S-wave in the first and second medium, respectively. The interface conditions that q and $\partial_x q$ be continuous across an interface at x = 0 can be written in terms of Q^{-1} and $\tilde{q}_o$, $\tilde{q}_R$ and $\tilde{q}_T$:

$$Q^{-1}\tilde{q}_o + Q^{-1}\tilde{q}_R = Q^{-1}\tilde{q}_T$$

$$\text{and} \quad Q^{-1}\partial_x\tilde{q}_o + Q^{-1}\partial_x\tilde{q}_R = Q^{-1}\partial_x\tilde{q}_T . \qquad (32)$$

Let us take as an example the variables $q = [\partial_x w, (1/\rho)\tau_{xz}]^T$. Q^{-1} is given by

$$Q^{-1} = \frac{1}{\omega_2}\begin{pmatrix} -\omega^2-2\beta_2\partial_{xx} & \partial_{xx} \\ -2\beta^2 & 1 \end{pmatrix}$$

Letting $a = [A, 0]^T$, $r = [R_p, R_s]^T$, and $\tau = [T_p, T_s]^T$, equations (32) become

$$a + r = Q_o^{-1}Q_1\tau$$

$$\text{and} \quad a - r = \Lambda_o^{-1}Q_o^{-1}Q_1\Lambda_1\tau \qquad (33)$$

where

$$Q_o = \begin{pmatrix} -\omega^2 + 2\beta^2k_{\alpha_1}^2 & -k_{\alpha_1}^2 \\ -2\beta^2 & 1 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} -\omega^2 + 2\beta^2k_{\alpha_2}^2 & -k_{\alpha_2}^2 \\ -2\beta^2 & 1 \end{pmatrix}$$

$$\Lambda_o = \begin{pmatrix} ik_{\alpha_1} & 0 \\ 0 & ik_{\beta_1} \end{pmatrix} \quad \text{and} \quad \Lambda_1 = \begin{pmatrix} ik_{\alpha_2} & 0 \\ 0 & ik_{\beta_2} \end{pmatrix}$$

Equations (33) can be solved explicitly for r and τ as a function of the incident angle $\theta = \cos^{-1}(\alpha_1 k_{\alpha_1}/\omega)$ and a. This was done numerically and the results are shown in figure 5. The biggest problem with these reflection and transmission coefficients is the singularity: the coefficients become very large for certain incident angles. This would seem

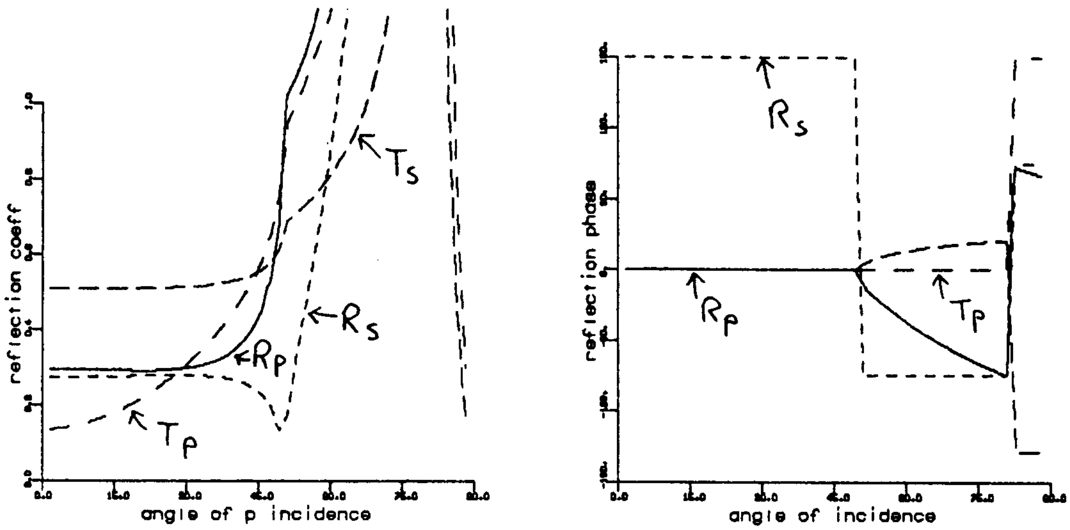

FIG. 5. The figure shows the apparent reflection and transmission coefficients, for the first set of mixed variables listed in table 1. The incident wave is a P wave. We believe the singularity in both the reflection and transmission coefficients gives rise to the instability shown in figure 4.

to insure instability. We tried all of the other sets of variables listed in the table and several other boundary conditions of the type

$$\Gamma_1 \partial_x q_1 = \Gamma_2 \partial_x q_2$$

where Γ_1 and Γ_2 are 2 by 2 matrices, but all of the ones tried exhibited the same features as the plot in figure 4. We concluded that the instability is probably unavoidable with these differential equations.

Conclusions

We have not been able to conclusively select a set of variables for elastic wave extrapolation. Each of the three types discussed fail to satisfy all of the requirements of an ideal operator. We feel, however, that for modeling purposes the displacements are the best choice. The present lack of higher order approximations limits the angular accuracy to a cone about the extrapolation direction. However, since the internal boundary conditions are satisfied the solutions should be correct within the cone.