COMPUTING LOVE WAVES IN LATERALLY VARYING MEDIA BY WAVE EXTRAPOLATION

Robert W. Clayton
David Brown

Introduction

One-way wave extrapolation equations provide an economical and accurate method for certain types of wave modeling. In this paper we consider the extrapolation with the scalar wave equations, and illustrate the method with the extrapolation of Love wave normal modes in laterally varying media.

The basic restriction on extrapolation modeling is that the problem contains no backscattered waves. This condition arises because waves moving backward to the direction of extrapolation are not included in the solution unless they are explicitly coupled with reflection coefficients. However, the method can be used to model refracted body waves, head waves, and surface waves.

The economy of extrapolation methods is essentially one of storage. Finite difference solutions of the full wave equation require storage proportional to the product of the x and z grid dimensions. The one-way equations, in their monochromatic form, only require storage proportional to one of the dimensions.

Modeling with one-way equations has a further advantage in that boundary conditions at layer interfaces need not be matched explicitly. If the correct form of the variable velocity one-way equation is used, then the boundary conditions can reasonably be approximated by simply varying the medium parameters.

Variable Velocity Extrapolation Equations

Consider the geometry for extrapolation shown in figure 1.

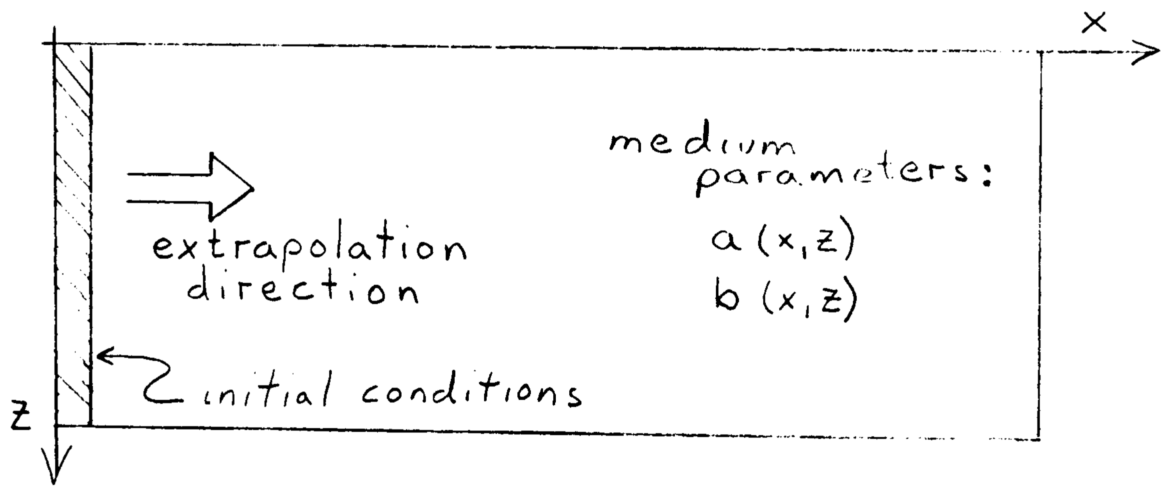

FIG. 1. The geometry for extrapolation modeling is shown. The initial conditions are specified on the left, and the solution in the rest of the medium is determined by extrapolation in the x-direction. The top surface is a free surface.

The medium is assumed to be governed by the scalar wave equation

$$(a\partial_z b\partial_z + ab\partial_{xx} - \omega^2)Q = 0 \tag{1}$$

where a and b are functions of x and z, and $ab = v^2$. If Q is a pressure variable (acoustic waves) then a is the bulk modulus and b is the inverse of density. If Q is displacement (SH waves) then a is the inverse of density and b is the shear modulus. The most general form of equation (1) would include the term $a(\partial_x b)\partial_x$ in the operator. This term corresponds to reflection and transmission coefficients in the direction of extrapolation, and in this paper it is omitted. The boundary conditions that accompany equation (1) are

$$[Q] = 0 \qquad \text{and} \qquad [b\hat{n}\cdot\nabla Q] = 0 \tag{2}$$

where $\hat{n}$ is the unit vector normal to the interface and the square brackets denote differences across the interface. For a horizontally layered medium the boundary conditions reduce to

$$[Q] = 0 \qquad \text{and} \qquad [b\partial_z Q] = 0 \ . \tag{3}$$

The one-way extrapolation equations are defined by the differential equation

$$(\partial_x - \frac{1}{v} S_{n+1})\ Q = 0 \tag{4}$$

where S_{n+1} is a square root approximation generated recursively by Muir's relation

$$S_{n+1} = \frac{i\omega S_n + (a\partial_z b\partial_z - \omega^2)}{i\omega + S_n} \ , \qquad S_0 = i\omega \tag{5}$$

The square root approximations given by equation (5) are well known to model accurately waves traveling within some cone of the extrapolation direction. The dispersion curves for the first and second approximations (15- and 45-degree) equations are shown in figure 2. The approximations have no evanescent zone in the k_z direction ($|k_z| > \omega/v$), but they do model behavior in the k_x evanescent zone ($|k_x| > \omega/v$). This is shown in the right panel of figure 2, which is simply a replotting of the dispersion relation to show its behavior for both real and imaginary values of k_x.

The square root operator in equation (5) will match the boundary conditions in (3) exactly for flat layers. This is because the z-differentials have the same form as the z-differentials in the full scalar wave equation. However, since the true boundary conditions are actually given by (2), the boundary conditions implied by the one-way

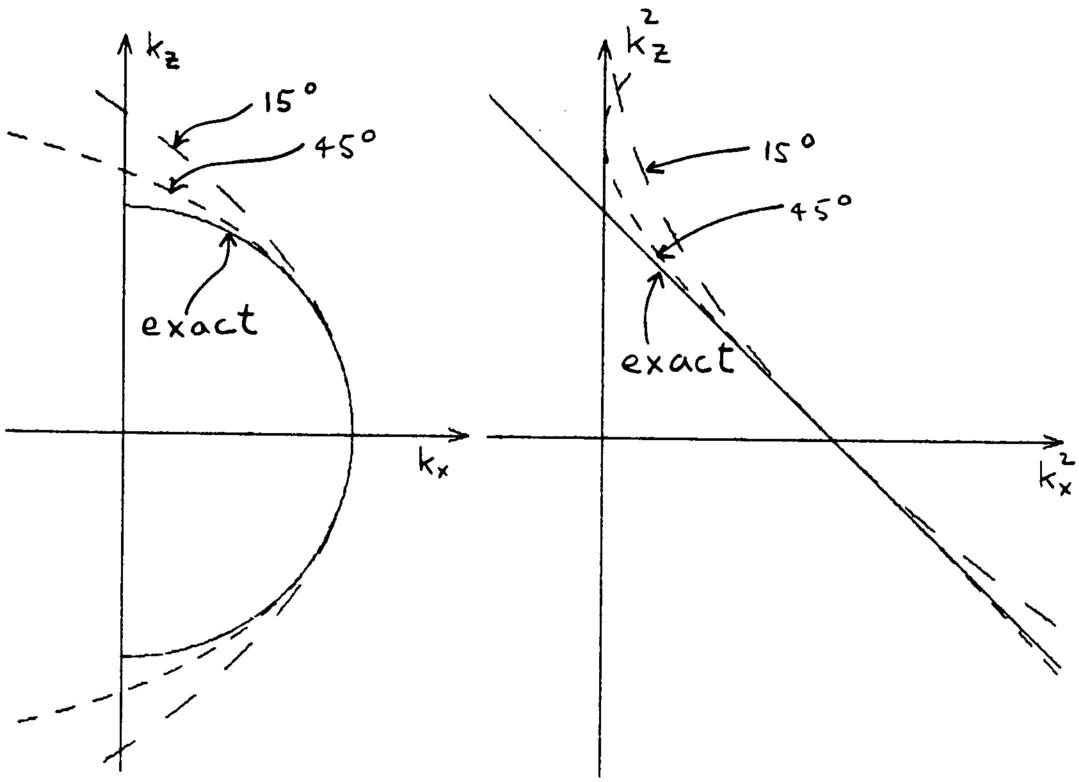

FIG. 2. The dispersion relations for the first two square root approximations (15- and 45-degree equations) are shown. The left panel shows the relations plotted in the conventional manner while the right panel shows the relations plotted in a manner which allows for imaginary values of k_z ($k_z^2 < 0$) and k_x ($k_x^2 < 0$). The exact dispersion relation is the straight line in the right plot. The extrapolation equations model waves well into the evanescent zone on the k_x-axis.

equations are approximate. In other words, the one-way equations use the boundary conditions given by (3), regardless of the orientation of the interface; whereas, if the modeling were exact the boundary conditions of (2) would be used.

The presence of a(x,z) outside the ∂_z operator in equation (5) means that we cannot guarantee unconditional stability of S_{n+1}. If, however, a is independent of z (or its dependence can be neglected) then $a\partial_z b\partial_z$ can be written as $\partial_z v^2 \partial_z$. This allows a finite difference representation of the z-differentials as

$$\partial_z v^2 \partial_z \simeq (vD_z)^T (vD_z)$$

which is an operator with strictly positive real eigenvalues. Following Muir's rules for causal positive real operators, the approximation S_{n+1} can be shown to be unconditionally stable. Thus, to guarantee the stability for displacement solutions, the density has to be independent of z; while for pressure solutions, the bulk modulus has to be independent of z. It is conjectured that the form of equation (5) will be stable for at least mildly varying values of a.

Love Wave Modes

As an example of extrapolation modeling we have computed Love wave mode shapes in simple laterally varying media. The example is useful for testing the modeling method because the modes contain both a propagating and an evanescent component. Also, because the analytic solution for horizontal layers is simple (the mode shape does not change), it is easy to check whether the program is producing the correct answer for the layered case.

As mentioned earlier, the SH-type displacements are governed by equation (1) with a as the inverse of density and b the shear modulus. Love waves (SH surface waves) have a solution to this equation of the form (see Achenbach, p. 218-220)

$$u(x,z,t) = U(z) \exp i\omega(t - \frac{x}{c}) \qquad (6)$$

where u(x,z,t) is the displacement in the y direction, U(z) is the mode shape, and c is the phase velocity of the mode. For a layered earth structure the boundary conditions to be satisfied are:

1) $\partial_z U = 0$ at z=0 (zero free surface stress),
2) $[U] = 0$ at layer boundaries (continuity of displacement),
3) $[\mu \partial_z U] = 0$ at layer boundaries (continuity of stress).

For the simple case of a layer of thickness h with properties ρ_1 and μ_1 over a half-space with properties ρ_2 and μ_2, the mode shape is given by

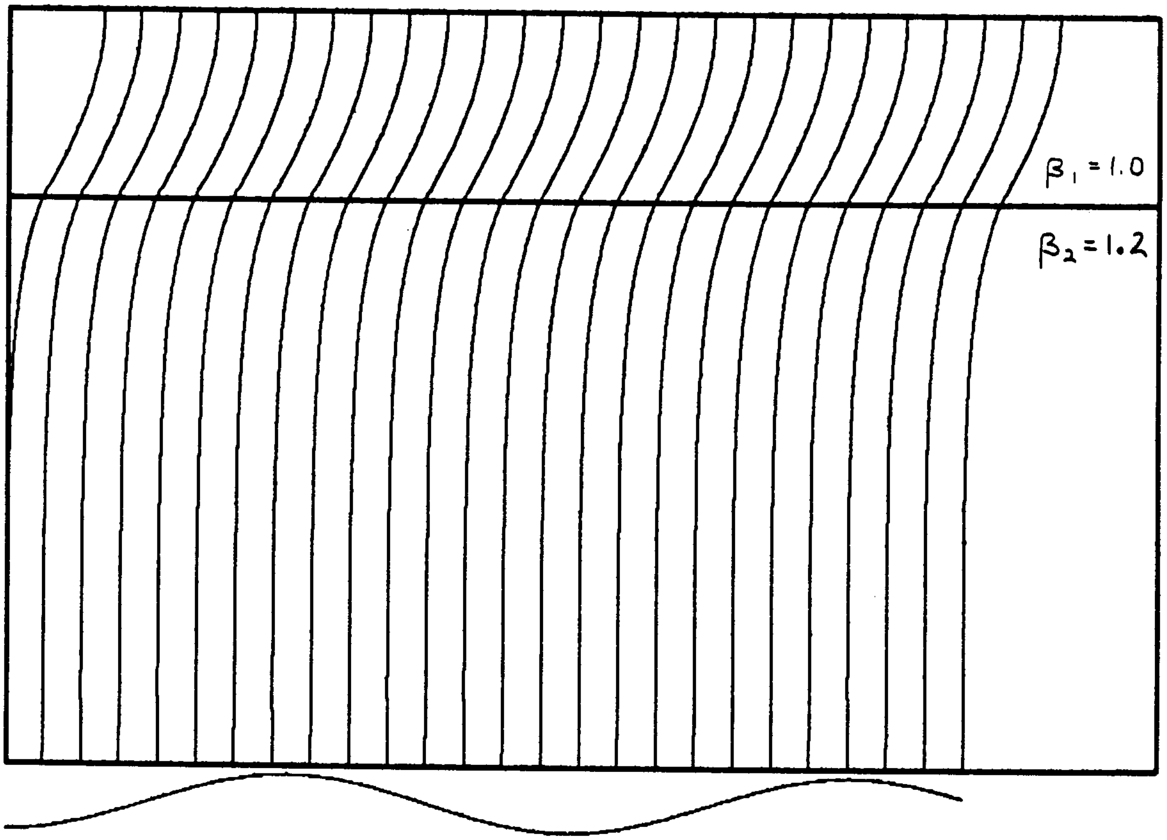

FIG. 3. The example shown is the extrapolation of a lower order Love wave mode for a layer over a half-space. The extrapolation is from left to right, with the initial mode shape shown on the left. The position of the layer is superimposed on the plot. The fact that the mode does not change shape significantly in the x-direction after demodulation indicates that the extrapolation solution is correct, and that the mode is being propagated at the correct phase velocity. The sinusoidal function plotted at the bottom is the demodulation function used. The frequency of the mode is 0.18 Hz and its speed is 1.04 km/sec. The plot has a 1 to 2 vertical exaggeration.

$$U(z) = \begin{cases} A\cos \nu_1 z & 0 \le z < h \\ A\cos \nu_1 h \exp[-i\nu_2(z-h)] & z > h \end{cases} \tag{7}$$

where

$$\nu_1 = \left(\frac{\omega^2}{\beta_1^2} - k_x^2\right)^{\frac{1}{2}} \quad \text{and} \quad \nu_2 = \left(\frac{\omega^2}{\beta_2^2} - k_x^2\right)^{\frac{1}{2}} . \tag{8}$$

The boundary conditions also impose the further restriction that

$$\cot(\nu_1 h) = \frac{\mu_1}{\mu_2}\frac{\nu_1}{i\nu_2} . \tag{9}$$

This equation is the period equation for the simplest type of Love waves, and it has solutions in the range

$$\frac{\omega}{\beta_1} > |k_x| > \frac{\omega}{\beta_2} .$$

With k_x in this range, the mode is propagating in the layer and evanescent in the half space. The period equation determines the relationship between ω and k_x, and since it is nonlinear, the modes are dispersive (velocity depends on frequency). The propagating speed of the mode is given by

$$c(\omega) = \frac{\omega}{k_x(\omega)} . \tag{10}$$

To test that the extrapolation works for a layered case, an initial mode shape was specified according to equation (7). The solution was then extrapolated in the x direction with a monochromatic 45 degree equation. The density was assumed constant. This allowed the guaranteed stable $\partial_z v^2 \partial_z$ form of the z differential to be used. At each step in the x direction the solution was multiplied before plotting by the demodulation factor

$$\exp\ i\frac{\omega}{c(\omega)}\, x$$

with $c(\omega)$ determined analytically from equation (10). This tests two aspects of the solution. First, multiplication by the demodulation factor effectively cancels the x-dependence of equation (6), leaving only the mode shape. Consequently, the solution if done correctly should be the same at all x-steps. Second, if the solution after multiplication by the demodulation factor remains invariant with respect to x, then it

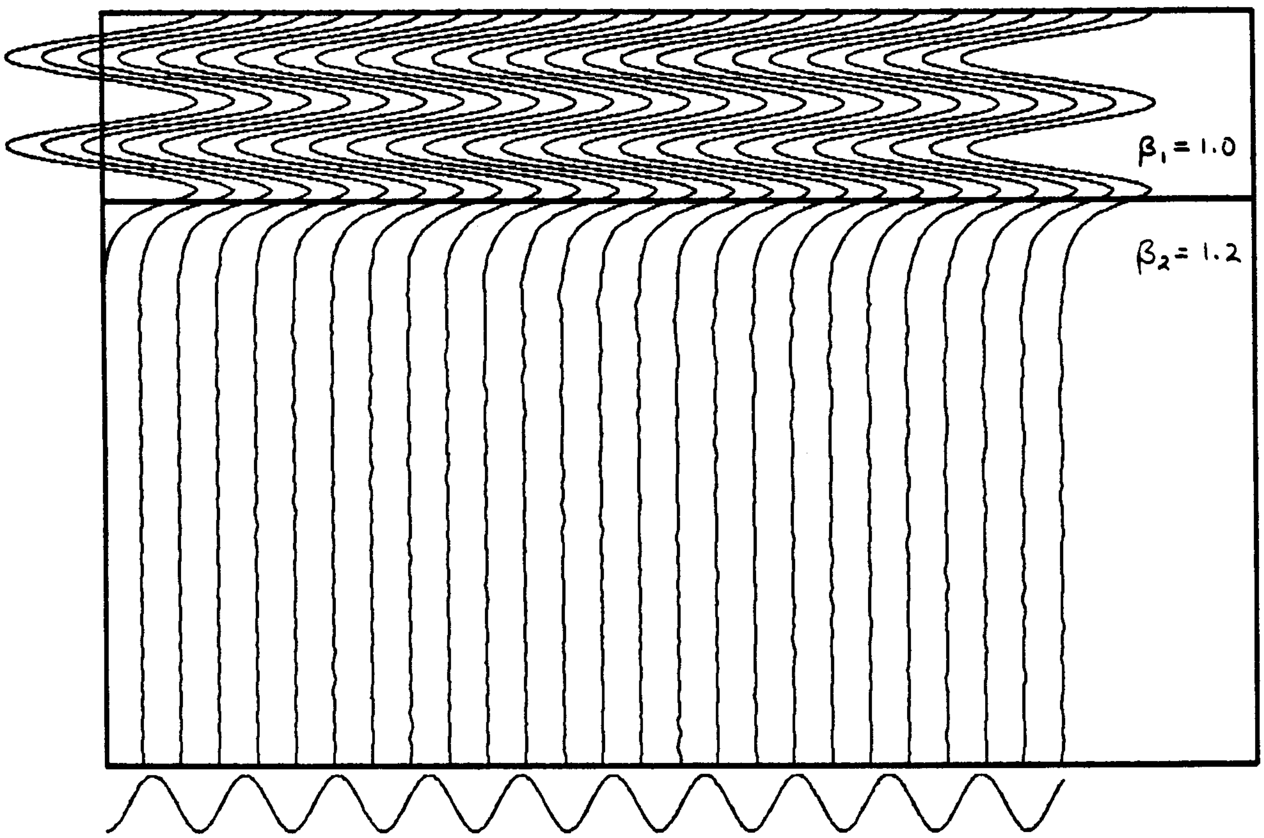

FIG. 4. This example is similar to the one shown in figure 3, except that the mode frequency is now 1.17 Hz., and its speed is 1.12 km/sec.

indicates that the mode is being propagated at the correct phase velocity. The results of propagating a low frequency and a high frequency mode through a layer over a half-space structure are shown in figures 3 and 4. The initial mode shape is plotted on the left in each figure and the solutions at various points in x appear to the right of it. The sinusoidal function displayed at the bottom of each figure is a plot of the demodulation factor.

The interesting case is when the model parameters vary laterally. For Love waves we have run two such cases. The first case is a dipping layer which dips down from the initial condition. The solution was again demodulated with a constant phase velocity determined from the initial conditions; but since the medium now varies laterally, this will not be sufficient to make the mode shape invariant with respect to x.

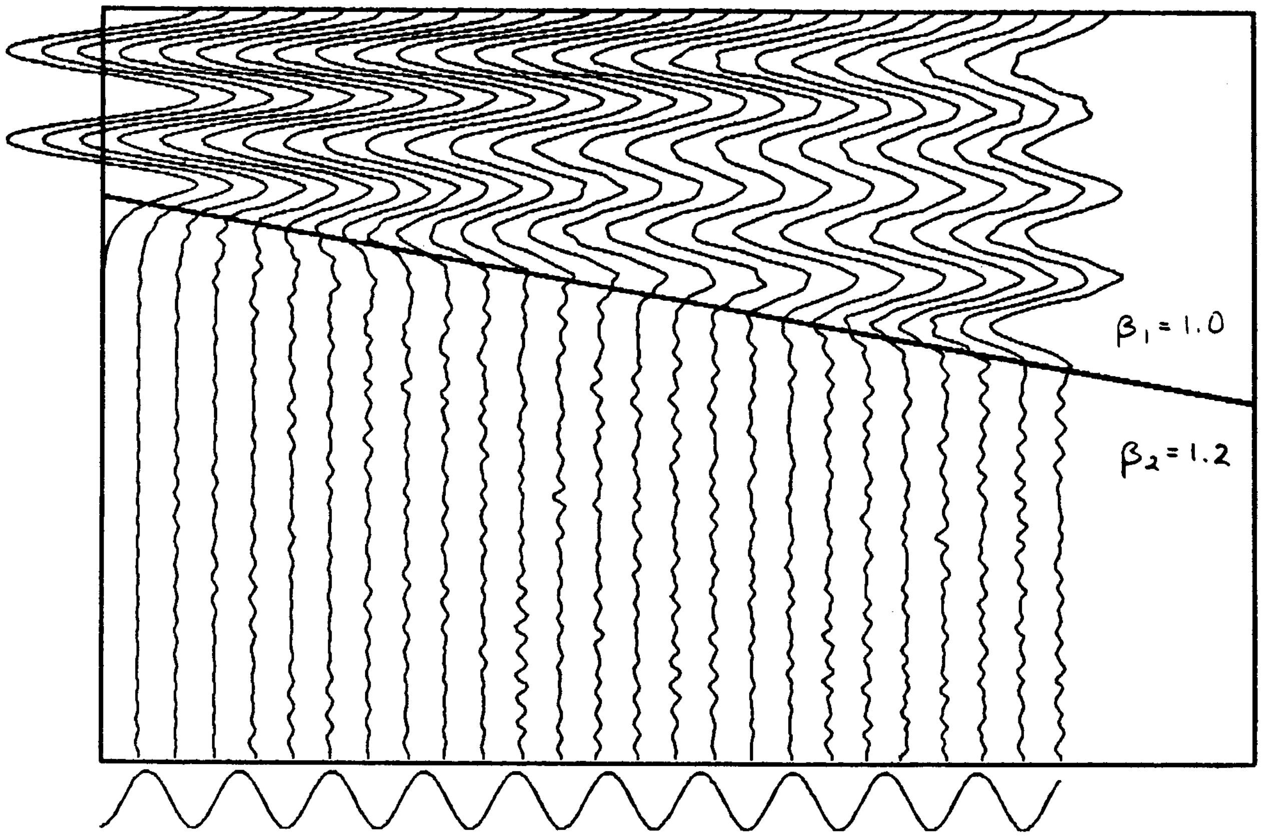

FIG. 5. A Love wave mode in a 20-degree down-dipping layer. The plot is similar to figure 3. The demodulation factor used is that of figure 3. The characteristic of the mode that seems to be preserved in the x-direction is that its spatial frequency in the layer is preserved. Also, very little energy is radiated into the half-space.

The second example is a dipping layer that dips up from the initial conditions. In this example, the energy contained within the layer is diminished and is radiated into the half-space in the form of SH body waves.

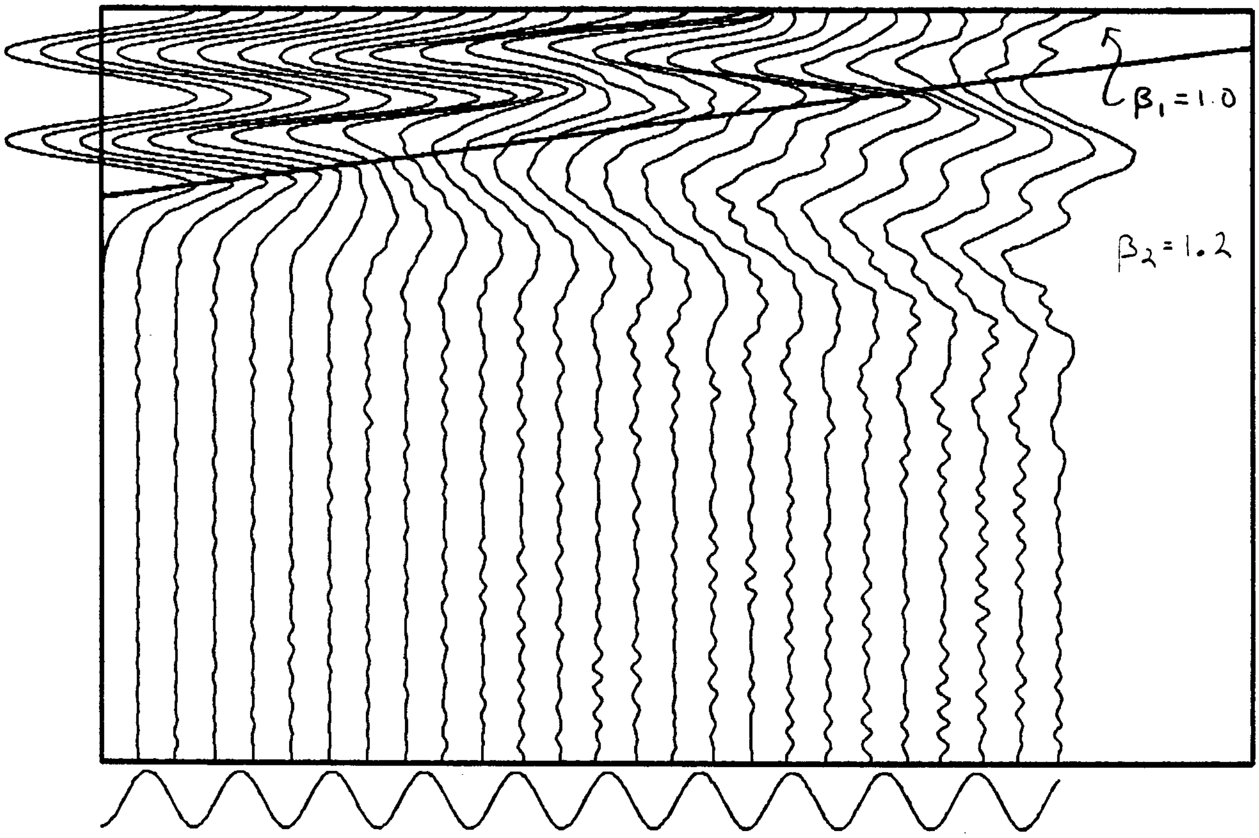

FIG. 6. In this example, the layer pinches out toward the surface. This time the mode radiates energy out of the layer into the half-space in the form of SH body waves.

REFERENCES

Achenbach, J.D., 1975, Wave propagation in elastic solids: Amsterdam, North-Holland.