A SCALAR MIGRATION EQUATION FOR CONVERTED SHEAR WAVES

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In previous SEP reports, downward-continuation equations have been given for elastic displacement fields. These equations simultaneously extrapolate both P and S waves, and hence could be used for a complete elastic imaging problem. They are, however, a rather complicated set of matrix equations, and perhaps more general than is really required. In this report, a much simpler scalar downward-continuation equation is derived for the specific problem of P waves converted to S waves at a reflector. Another part of the problem, that of separating the recorded displacement fields into P and S waves, is also discussed.

Consider the model of the earth shown in Figure 1. A compressional source (dynamite perhaps) is set off and radiates P waves downward. Both P waves and converted S waves are back-scattered at the reflectors and are recorded by a linear array of two component geophones. Mathematically, the reflectors are defined by the reflectivity functions R_{PP} for P wave reflections, and R_{PS} for converted S wave reflections. In this model the transmitted wave (the downgoing P wave) is unaffected by the reflectors. This is the same assumption that is used in acoustic migration theory, but in this case it has the added effect of disregarding S wave conversions on transmission. The transmission conversions are a higher order effect than reflection conversions, because they must undergo an additional reflection to be seen at the surface. Variable P and S velocities in this model serve only to bend the rays and do not contribute to the reflectivity.

To find the downward-continuation operator, we will take the same approach as with non-zero-offset migration (see Clayton, SEP-14).

First, the wavefield as a function of geophone position is extrapolated to a datum z by the equation

$$S(k_g,k_s,z_g{=}z,z_s{=}0,\omega) \;=\; \exp\left[-i\left(\frac{\omega^2}{\beta^2} - k_g^{\,2}\right)^{1/2}\right]z \;\; S(k_g,k_s,z_g{=}0,z_s{=}0,\omega) \qquad (1)$$

The velocity in the operator is the shear wave velocity β, since the upcoming wave to the geophones is the converted S wave. Next, the wavefield as a function of shot position is downward-continued according to

$$S(k_g,k_s,z_g{=}z_s{=}z,\omega) \;=\; \exp\left[-i\left(\frac{\omega^2}{\alpha^2} - k_s^{\,2}\right)^{1/2}\right]z \;\; S(k_y,k_s,z_g{=}z,z_s{=}0,\omega) \qquad (2)$$

Here the P wave velocity α is used because the downgoing wave is assumed to be compressional. Combining Equations (1) and (2) leads to the complete downward-continuation operator

$$S(k_g,k_s,z,\omega) \;=\; \exp[-i\;\Phi(k_g,k_s,\omega)z] \;\; S(k_g,k_s,z{=}0,\omega) \qquad (3)$$

where

$$\Phi(k_g,k_s,\omega) \;=\; \left(\frac{\omega^2}{\beta^2} - k_g^{\,2}\right)^{1/2} + \left(\frac{\omega^2}{\alpha^2} - k_s^{\,2}\right)^{1/2} \qquad (4)$$

We now convert to midpoint (y)- and half-offset (h)-coordinates with the wavenumber transformation

$$k_y \;=\; \frac{k_y + k_h}{2} \qquad\qquad k_s \;=\; \frac{k_y - k_h}{2} \qquad (5)$$

Now the downward continuation operator is

$$S(k_y,k_n,z,\omega) = \exp[-i\,\Psi(k_y,k_n,\omega)z]\ S(k_y,k_n,0,\omega) \tag{6}$$

where

$$\Psi(k_y,k_n,\omega) = \left[\frac{\omega^2}{\beta^2} - \left(\frac{k_y + k_n}{2}\right)^2\right]^{1/2} + \left[\frac{\omega^2}{\alpha^2} - \left(\frac{k_y - k_n}{2}\right)^2\right]^{1/2} \tag{7}$$

The imaging principle remains the same as for the acoustic case. That is, reflectors are imaged at $t = 0$ and $h = 0$, or

$$R_{PS}(y,z) = \lim_{\substack{h\to 0\\ t\to 0}} S(y,h,z,t) \tag{8}$$

The variation of R_{PS} with incident angle is more pronounced than R_{PP}, and the above imaging scheme should produce an average of R_{PS} as a function of angle.

To check that the operator is reasonable, we can perform a stationary phase approximation on the inverse Fourier transform (Clayton, SEP-14). This gives the shape of the impulse response of the operator in (y,h,z,t) space as

$$t = \frac{1}{\alpha}[z^2 + (y - h)^2]^{1/2} + \frac{1}{\beta}[z^2 + (y + h)^2]^{1/2} \tag{9}$$

which agrees with the ray theory traveltimes shown in Figure 2.

The operator has the form of the double square root equation of acoustic migration, differing only in the fact that the velocities in each square root differ. Its behavior, however, is significantly different. For example, the CMP stacking operator $(k_y = 0)$ is

$$\Psi(0,k_h,\omega) = \left(\frac{\omega^2}{\beta^2} - \frac{1}{4}k_h^{\,2}\right)^{1/2} + \left(\frac{\omega^2}{\alpha^2} - \frac{1}{4}k_h^{\,2}\right)^{1/2} \tag{10}$$

which is more complicated than the single square root of the scalar case. A stationary phase approximation of this operator leads to the parametric equations relating t and offset (h):

$$\begin{aligned} t(p) &= \frac{z}{\alpha}(1 - p^2\alpha^2)^{-1/2} + \frac{z}{\beta}(1 - p^2\beta^2)^{-1/2} \\ h(p) &= zp\alpha(1 - p^2\alpha^2)^{-1/2} + zp\beta(1 - p^2\beta^2)^{-1/2} \end{aligned} \tag{11}$$

where p is Snell's parameter. The last equation is a quartic in p, which is sufficient grounds to abandon the attempt to find the exact stacking trajectory in h-t space. The complicated form of Equations (11) arise because the reflection points for converted shear waves are not under the midpoints even for a layered earth. A second order Taylor series expansion of the stacking operator is

$$\Psi(0,k_h,\omega) = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{1}{\omega} - \frac{1}{8}(\alpha + \beta)\frac{k_h^2}{\omega} \tag{12}$$

This shows that the vertical traveltime (t_0), which is the first term in the expansion, is controlled by the average slowness $(1/\alpha + 1/\beta)$. The second term or normal moveout is controlled by the depth average of the velocities $(\alpha + \beta)$.

The zero-offset $(k_h = 0)$ migration operator has similar complications and interpretations. The diffraction term is controlled by the average velocity, while the shifting term is controlled by the average slowness. The various schemes used to approximate the double square root equation (Clayton, SEP-15; Yilmaz, this report) in scalar theory have not as yet been applied to Equation (7). The complication of the reflection points not lying under the midpoints may indicate that a new coordinate system in the (x,y,h,t) domain may be useful.

To apply the operator derived above, it is necessary to separate the upcoming P wave from the converted S wave. A 5-degree method (in the same sense that the 15-degree equation is accurate to 15°) is simply to assume that vertical geophone measures p waves, and the horizontal geophone measures S waves.

This is an approximation of an exact method (in theory), which can be obtained from the definitions P and S.

$$\underline{u} = \cdot\ P + \nabla \times S \qquad \underline{u} = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \text{horizontal displ} \\ \text{vertical displ.} \end{pmatrix}$$

or (13)

$$u = P_x - S_z$$

$$w = P_z - S_x$$

Now the S field moves at velocity β; hence,

$$S_z = i\left(\frac{\omega^2}{\beta^2} - k_x{}^2\right)^{1/2} S \equiv i\, Q_\beta\, S \tag{14}$$

Similarly, $P_z = i\, Q_\alpha\, P$. Therefore, in the $k_x - \omega$ domain,

$$\begin{pmatrix} u \\ w \end{pmatrix} = i \begin{pmatrix} k_x & -Q_\beta \\ Q_\alpha & k_x \end{pmatrix} \begin{pmatrix} P \\ S \end{pmatrix} \tag{15}$$

Inverting the equations, we have

$$\begin{pmatrix} P \\ S \end{pmatrix} = \frac{-i}{k_x{}^2 + Q_\alpha Q_\beta} \begin{pmatrix} k_x & Q_\beta \\ Q_\alpha & k_x \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \tag{16}$$

For vertical incidence, the operator is

$$\begin{pmatrix} P \\ S \end{pmatrix} = \frac{-i}{\omega} \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \tag{17}$$

which is the 5-degree approximation mentioned above. It is somewhat annoying to have the P and S fields integrated with respect to time. This can be removed by simply multiplying the right side of Equation (14) by ω.

The exact conversion scheme was tested on a simple example shown in Figure 3. Displayed are the synthetic zero-offset displacement fields of a point scatter, with an incident P wave. The scatter (unphysically) reflects converted S waves with the same strength as P waves. The purpose of the example is simply to test the separability of P and S waves. The first arrival in both the horizontal and vertical displacement fields is the reflected P wave. The hyperbola is symmetric in vertical displacement and anti-symmetric in horizontal displacement. The second arrival in both fields is the converted shear wave (P down, S up), and it has the opposite symmetry properties of the P wave. In Figure 4 the same results of applying the operator of Equation (14) (actually ω times the operator) are shown. The operator as succeeded in separating the P and S waves. It is believed that the errors that do occur in the separation (the wraparound, for example) are due to the truncation of the hyperbolae in the synthetic data.

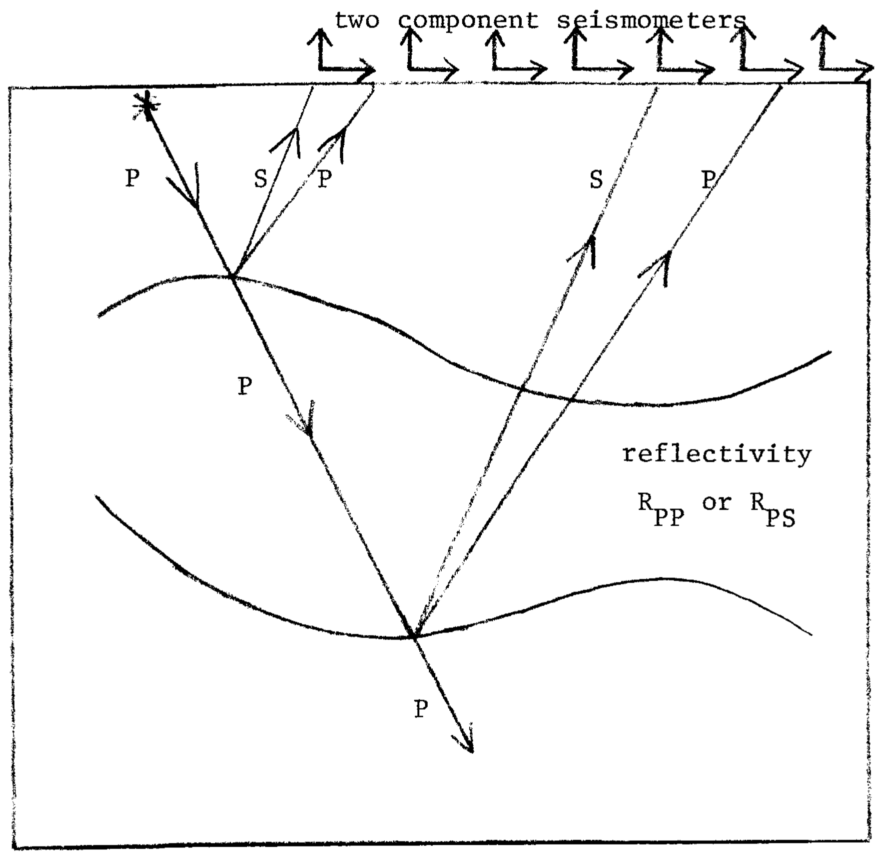

Figure 1. Model of the Earth For Converted Shear Waves
The downgoing P wave is backscattered at each reflector according to the reflectivity functions R_{PP} and R_{PS}. The recording devices are two component seismometers. The reflectors do not alter the downgoing P wave and do not initiate a transmitted S wave.

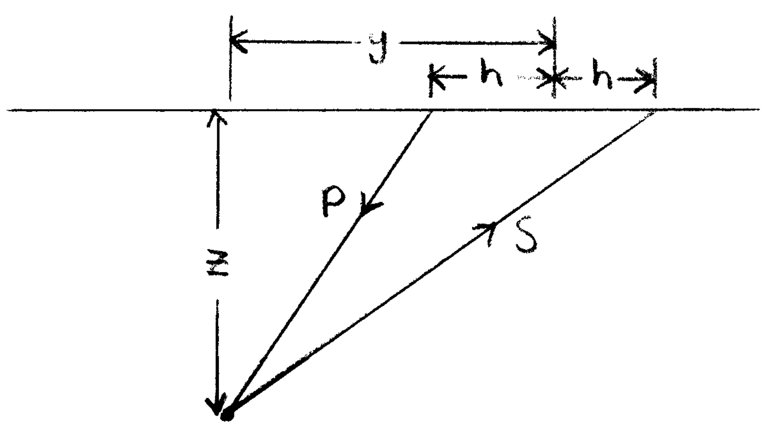

Figure 2. Ray Diagram For Converted Shear Waves Over a Point Scatter The ray theory travel time for a converted S wave given in equation (9) can be verified by a simple appication of geometry to this figure.

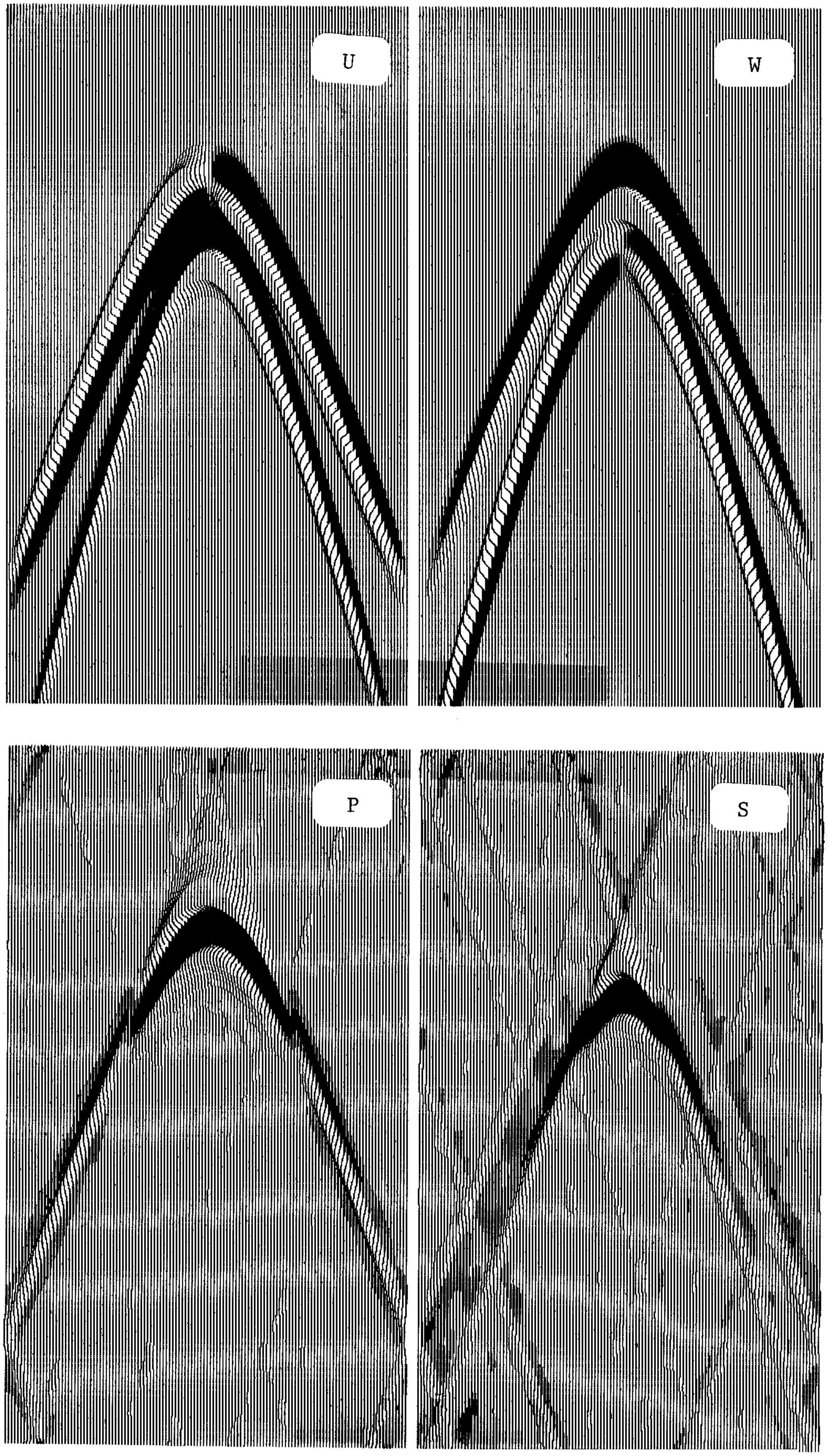

Figure 3. Conversion Of Displacements To P and S Waves
In the upper panel, the horizontal (u) and vertical (w) displacement fields are shown. The results of applying the conversion operator are shown in the lower panel. The errors in conversion are believed to be truncation effects.