BUSSGANG PROCESSES

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Definition of Bussgang Process

In the early 1950's, Bussgang [see Barrett and Lampard (1955)], demonstrated an interesting and useful result concerning noise and non-linear devices. He showed that for Gaussian processes, the cross-correlation (C/C) between the input and output, obtained by passing a time series through a zero-memory non-linear (ZNL) device, was proportional to the autocorrelation (A/C) of the input. A practical application of this result is computing the A/C by correlating a time series with the sgn of itself. No multiplications are involved; the variance of the time series, however, must be independently computed. With a view of extending this result to processes other than Gaussian, Barrett and Lampard defined a new class of distributions called the Λ class, which also had the "Bussgang" property. The general telegraph matrix P_t ("Modeling Seismic Impedance by Markov Chains - Model Properties," this report) is in fact a member of the Λ-class, and although Bussgang, the autocorrelation function is strictly exponential. For Gaussian processes the A/C function is arbitrary. Are there any other useful distributions or probability transition matrices that are not Gaussian, do not have an exponential A/C, and are still Bussgang?

The class of Bussgang Processes

To answer the question posed above, we follow Barrett and Lampard and expand a general second order stationary probability density function (pdf) into a double Fourier series involving one set of normalized orthogonal polynomials:

$$f_{x_1,x_2}(x_1,x_2;t) = f_x(x_1)\, f_x(x_2) \sum_{m,n=0}^{\infty} a_{mn}(t)\, o_m(x_1)\, o_n(x_2) \quad \text{(1a)}$$

$f_{x_1,x_2}(x_1,x_2;t)$ = second order pdf whose random variables (RV) X_1 and X_2 are members of the process $\{X_t\}$ separated by t units.

$f_x(x)$ = first-order pdf. All RV's in $\{X_t\}$ have identical first-order pdf.

o_j = orthogonal [with respect to $f_x(x)$] polynomials:

$$\int f_x(x_1)\, o_m(x_1)\, o_n(x_1)\, dx_1 = \delta_{mn} \quad \text{(1b)}$$

$a_{mn}(t)$ = Fourier coefficients

$$= \iint f_{x_1,x_2}(x_1,x_2;t)\, o_m(x_1)\, o_n(x_2)\, dx_1\, dx_2$$

Theorem: A second-order stationary stochastic process is Bussgang if and only if:

$$E[X_1|X_2=x_2;t] = p(t)\, h(x_2) \quad \text{(2)}$$

$p(t)$ = autocorrelation of $\{X_t\}$

$h(\cdot)$ = arbitrary function

Corollary: The process is Bussgang if and only if the Fourier coefficients have the minimal restriction:

$$a_{1j}(t) = a_{j1}(t) = 0 \text{ or } a_{11}(t),\ j \neq 1 \quad \text{(3)}$$

Proof

Assume that $\{X_t\}$ is zero-mean, unit variance.

(1) For $\{X_t\}$ to be Bussgang, the C/C must be symmetric:

$$E[X_1 g(X_2);t] = E[X_2 g(X_1);t]$$

$$g = \text{ZNL}$$

But $E[X_1 g(X_2);t] = \iint x_1 g(x_2)\, f_{x_1,x_2}(x_1,x_2;t)\, dx_1 dx_2$ (4)

Making use of the expansion (1a), Equation (4) becomes

$$E[X_1 g(X_2);t] = \sum_{mn} a_{mn}(t) \int g(x_2) f_x(x_2) o_n(x_2) \left[\int x_1 f_x(x_1) o_m(x_1) dx_1 \quad dx_2\right] \tag{5}$$

Before proceeding, we define the first two polynomials:

$$o_0(x) = 1 \tag{6a}$$

$$o_1(x) = x \tag{6b}$$

Both (6a) and (6b) are consistent with Equation (1b). Since the o_j are mutually orthogonal, the inner integral in Equation (5) exists only when $m = 1$, and we obtain

$$E[X_1 g(X_2);t] = \sum_{n} a_{1n}(t) \int g(x_2) f_x(x_2) o_n(x_2) dx_2 \tag{7a}$$

Similarly,

$$E[X_2 g(X_1);t] = \sum_{m} a_{m1}(t) \int g(x_1) f_x(x_1) o_m(x_1) dx_1 \tag{7b}$$

Equation (7a) and (7b) are equal if and only if:

$$a_{1q} = a_{q1}, \quad q \geq 0 \tag{8}$$

(2) Next, determine the functional form of $R_{xx}(t)$

$$R_{xx}(t) = E[X_1 X_2;t]$$

$$= \iint x_1 x_2 f_{x_1,x_2}(x_1,x_2;t) dx_1 dx_2$$

If the expansion in Equation (1a) is substituted into the above equation, we eventually find that

$$R_{xx}(t) = a_{11}(t) \triangleq p(t) \tag{9}$$

(3) Third, determine the crosscorrelation, $R_{xg(x)}(t)$ where g is a ZNL:

$$R_{xg(x)}(t) = E[X_1 g(X_2);t]$$
$$= \iint x_1 g(x_2)\, f_{x_1,x_2}(x_1,x_2;t)dx_1 dx_2$$
$$= \int g(x_2)\left[\int x_1 f_{x_1|x_2}(x_1|x_2;t)dx_1\right] f_x(x_2)dx_2$$

The term in brackets is the conditional expectation:

$$R_{xg(x)}(t) = \int g(x_2)\ E[X_1|X_2{=}x_2;t]\ f_x(x_2)dx_2 \tag{10}$$

(4) Bussgang property holds if and only if the ratio of Equations (9) and (10) is constant:

$$\frac{R_{xg(x)}(t)}{R_{xx}(t)} = \text{constant if and only if:}$$

$$E[X_1|X_2{=}x_2;t] = a_{11}(t)\ h(x_2) \tag{11}$$

where $h(\cdot)$ is an arbitrary function.

(5) Equation (11) can be simplified if the conditional expectation is expanded:

$$E[X_1|X_2{=}x_2;t] = \frac{\int x_1 f_{x_1,x_2}(x_1,x_2;t)dx_1}{f_x(x_2)}$$
$$= \sum_n a_{1n} o_n(x_2) \tag{12}$$

Equating Equations (11) and (12), we have

$$a_{1j} = 0 \text{ or } a_{11}\ ,\ j \neq 1 \tag{13}$$

We note the following relation (see *Examples*):

$$a_{1j} = 0\ ,\ j \neq 1 \Longleftrightarrow E[X_1|X_2 = x_2;t] = a_{11}(t)\ x_2$$

The corollary is obtained by using Equations (13) and (8).

Although the theorem is quite tedious to prove, it can be used to readily determine whether a process is Bussgang.

Examples

(1) A process having a second order Gaussian pdf of arbitrary correlation $p(t)$ is Bussgang. The theorem can be used to give a one-line proof. Write down the well-known formula for conditional density:

$$f_{X_1|X_2}(x_1|x_2;t) = \frac{1}{\sqrt{2\pi\sigma^2[1-p(t)^2]}} \exp\left[\frac{-[x_1 - p(t)x_2]^2}{2\sigma^2[1-p(t)^2]}\right]$$

from which $E[(X_1|X_2=x_2;t] = p(t)x_2$ is readily obtained. QED.

(2) This example is from "Modeling Seismic Impedance...", where it was shown that processes synthesized using the general telegraph matrix P_T are Bussgang. An alternative proof of this result makes use of the above theorem. First, expand P_T in terms of its eigenvectors:

$$P_T^k = \sum_{i=1}^{M} \lambda_i^k \mu_i \beta_i^T \tag{14}$$

μ_i = column eigenvector

β_i^T = row eigenvector

Since $\lambda_1 = 1$ and $\lambda_i = \lambda$, $i \geq 2$, Equation (14) becomes

$$P_T^k = \Gamma \alpha^T + \lambda^k \sum_2^M \mu_i \beta_i^T \tag{15}$$

Γ = column vector of ones

α = probability mass function

But $$I = \sum_1^M \mu_i \beta_i^T = \Gamma \alpha^T + \sum_2^M \mu_i \beta_i^T \tag{16}$$

Using (16), Equation (15) becomes

$$P_T^k = \Gamma \alpha^T + \lambda^k (I - \Gamma \alpha^T)$$

$$= \lambda^k I + (1 - \lambda^k) \Gamma \alpha^T$$

$$\text{or} \quad \{P_T^k\}_{ij} = \lambda^k \delta_{ij} + (1 - \lambda^k) \Gamma_i \alpha_j \tag{17}$$

Next, calculate the conditional expectation:

$$E[X_1 | X_2 = x_i ; k] = \sum_j x_j P_r [Q_{N+k} = x_j | Q_N = x_i]$$

$$= \sum_j x_j \{P_T^k\}_{ij} \tag{18}$$

Substituting Equation (17) into (18) and noting that $\alpha^T x = 0$, gives

$$E[X_1 | X_2 = x_i ; k] = \sum_j x_j \lambda^k \delta_{ij} + (1 - \lambda^k) \sum_j \Gamma_i x_j \alpha_j$$

$$= \lambda^k x_i \quad \therefore \quad \text{Bussgang}$$

REFERENCE

BARRETT, B.F. and D.G. LAMPARD, "An expansion for some second-order probability distributions and its applications to noise problems," *IRE Transactions - Information Theory*, March 1955.