VELOCITY ANISOTROPY

Bert Jacobs

Abstract

Elastic wave velocity anisotropy in rocks is easy to model if the elements of the stiffness matrix satisfy a constraint equation. The effects of the simplest kind of anisotropy on depth sections can then be accounted for by a simple stretch of the z-axis.

Introduction

Velocity analyses of seismic data measure the speed with which sound travels in the horizontal direction. Well logs measure the speed of sound that travels in a vertical direction. There is often a discrepancy between the two measurements and this implies an ambiguity in the time-depth conversion of seismic sections. One reason for this discrepancy is that rocks often exhibit anisotropies of up to 10 percent.

The easiest kind of anisotropy to model is that which discriminates in azimuth alone but which does not differentiate between angles in a horizontal plane. It is probable that most geological materials exhibit either this kind of anisotropy, called hexagonal in Auld's text, or are isotropic.

Christoffel equations

The frequency domain propagation equations for elastic waves in homogeneous materials are called the Christoffel equations of medium. They are expressed in terms of the wave-number vector (k_x, k_y, k_z) pointing in the direction of propagation, the particle displacements

(u_x, u_y, u_z), the stiffness matrix coefficients c_{ij} $(i,j = 1,2,\ldots,6)$, and the material's density ρ. Since our system is hexagonal it is rotationally symmetric about any vertical axis. It follows that we can, without loss of generality, set $k_y = 0$. The Christoffel equations are then

$$\begin{bmatrix} c_{11}k_x^2 + c_{44}k_z^2 & 0 & c_{13} + c_{44}k_xk_z \\ 0 & c_{66}k_x^2 + c_{44}k_z^2 & 0 \\ (c_{13} + c_{44})k_xk_z & 0 & c_{44}k_x^2 + c_{33}k_z^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u \end{bmatrix} = \rho\omega^2 \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

with the hexagonal constraint that $c_{66} = 1/2(c_{11} + c_{33})$.

If the Christoffel matrix is nonsingular, then the unique solution of the system of equations is the null-vector. A non-trivial solution is obtained only when the determinant of the matrix vanishes, so we are led to consider the eigenvalues and vectors of the matrix. The equations for the eigenvalues will be the dispersion relations for the propagating wavefields. There will be three such eigenmodes - in an elastic medium these correspond to a P-wave and two S-waves. In the hexagonally symmetric medium we are studying, the eigenmodes are called pseudo-P, pseudo-S, and SH. Defining auxiliary variables A and B

$$A = \frac{1}{2}\left[(c_{11} - c_{44})k_x^2 + (c_{44} + c_{33})k_z^2\right]$$

$$B = \frac{1}{2}\sqrt{\left[(c_{11} - c_{44})k_x^2 + (c_{44} - c_{33})k_z^2\right]^2 + 4(c_{13} + c_{44})^2k_x^2k_z^2}$$

these equations are

$\rho\omega^2 = A + B$		Pseudo-P
$\rho\omega^2 = A - B$		Pseudo-S
$\rho\omega^2 = c_{66}k_x^2 + c_{44}k_z^2$		SH

The square roots in these equations are a nuisance for computational purposes. They will disappear if the stiffness tensor components satisfy an additional constraint, changing the dispersion relations to

$$\rho\omega^2 = c_{11}k_x^2 + c_{33}k_z^2 \qquad \text{Pseudo-P}$$

$$\rho\omega^2 = c_{44}k_x^2 + c_{44}k_z^2 \qquad \text{Pseudo-S}$$

$$\rho\omega^2 = c_{66}k_x^2 + c_{44}k_z^2 \qquad \text{SH}$$

$$c_{13} = -c_{44} + (c_{11} - c_{44})(c_{33} - c_{44}) \qquad \text{Constraint}$$

There are three eigenvectors corresponding to these three eigenmodes.

$$U_P = \begin{bmatrix} k_x \\ 0 \\ Ck_z \end{bmatrix} \qquad U_S = \begin{bmatrix} k_x \\ 0 \\ -Ck_z \end{bmatrix} \qquad U_{SH} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where the constant C is defined by

$$C = \sqrt{\frac{c_{33} - c_{44}}{c_{11} - c_{44}}}$$

Implementation

The time domain analogue of the pseudo-P dispersion relation is a wave equation which governs the diffraction of waves much like the P waves observed in isotropic materials. The partial differential equation is

$$D_{tt}u = v_{11}^2 D_{xx}u + v_{33}^2 D_{zz}u$$

where v_{11} is the velocity of sound traveling in a horizontal direction and v_{33} is the velocity when the wave travels vertically. These two velocities are functions of the stiffness coefficients and the material density.

$$v_{11} = \sqrt{\frac{c_{11}}{\rho}} \qquad\qquad v_{33} = \sqrt{\frac{c_{33}}{\rho}}$$

The anisotropic wave equation can be derived from the isotropic equation by a simple change of variables corresponding to a simple stretch of the z-axis. This means that if anisotropic effects are to be included in a migration, then the only change in current procedures is a stretch made after the migration is done. For a z-variable medium and a migration done with moveout velocities $v_{11}(z)$, the post-migration stretch is given by the equation

$$z' = \int_0^z \frac{v_{33}}{v_{11}}\, ds$$

Rock physics

The reasonableness of the constraint on the stiffness matrix that simplifies the Christoffel equations is discussed in a paper by Nur. He derived it four years earlier by considering the closing of cracks in rocks due to vertical loading.

REFERENCES

AULD, B. A. (1973). *Acoustic Fields and Waves in Solids, Vol. 1* (John Wiley & Sons).

NUR, A. (1971), "Effects of Stress on Velocity Anisotropy in Rocks with Cracks," *Journal of Geophysical Research,* 76, pp. 2022-34.

APPENDIX

Let the components of the strain tensor be defined so that e_{ij} is the partial derivative of u_j in the i-direction, where u is a vector displacement. The indices i and j run over x, y, and z. Similarly, denote the stress in the i-direction on the j-face of a small element of material by I_j, where I varies over X, Y, and Z.

The elements of the stiffness matrix are defined by and where j ranges over x, y, and z.

$$X_x = c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}e_{yz} + c_{15}e_{zx} + c_{16}e_{xy}$$

$$Y_y = c_{21}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz} + c_{24}e_{yz} + c_{25}e_{zx} + c_{26}e_{xy}$$

$$Z_z = c_{31}e_{xx} + c_{32}e_{yy} + c_{33}e_{zz} + c_{34}e_{yz} + c_{35}e_{zx} + c_{36}e_{xy}$$

$$Y_z = c_{41}e_{xx} + c_{42}e_{yy} + c_{43}e_{zz} + c_{44}e_{yz} + c_{45}e_{zx} + c_{46}e_{xy}$$

$$Z_x = c_{51}e_{xx} + c_{52}e_{yy} + c_{53}e_{zz} + c_{54}e_{yz} + c_{55}e_{zx} + c_{56}e_{xy}$$

$$X_y = c_{61}e_{xx} + c_{62}e_{yy} + c_{63}e_{zz} + c_{64}e_{yz} + c_{65}e_{zx} + c_{66}e_{xy}$$