POWERS OF CAUSAL OPERATORS

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Abstract

A minimum phase function can be raised to any power and the result is minimum phase. An impedance function can be raised to any fractional power ρ where $-1 \leq \rho \leq +1$ and the result is an impedance function. The causal integration operator is an impedance function. The exact square root downward extrapolation operator is an impedance function. Its square is a minimum phase function that exhibits "branch cut" behavior. A fractional power of the causal integration operator gives a constant Q form of Hooke's law. The logarithm in the frequency domain of the causal integration operator is one side of the Hilbert transform in the time domain. Physically this waveshape arises as the reflection of an impulse from an interface between two media of two different constant Q's.

Introduction

Powers of causal operators arise naturally in problems of wave propagation, extrapolation, and dissipation. To facilitate both comprehension and computation, some basic "functional analysis" theorems will first be developed with Z-transforms. Then they will be applied to examples of migration and constant Q dissipation.

Functional analysis

We will establish, in sequence, the following theorems about exponentials, logarithms and powers of Fourier transforms of filters:

1. The exponential of a causal filter is causal.

2. The exponential of a causal filter is minimum phase.

3. The logarithm of a minimum phase filter is causal.

4. Any real power of a minimum phase filter is minimum phase.

5. Any fractional power $-1 \leq \rho \leq 1$ of an impedance function is an impedance function.

To establish Theorem 1 we define the Z-transform of an arbitrary causal function

$$U(Z) = u_0 + u_1 Z + u_2 Z^2 + \ldots \tag{1}$$

and substitute it into the familiar power series for exponential

$$B(Z) = e^U = 1 + U + \frac{U^2}{2} + \ldots \qquad (|U| < \infty) \tag{2}$$

It is clear that no negative powers of Z will be generated so that B(Z) is also causal.

To establish Theorem 2, that the exponential is not just causal but also minimum phase, we consider

$$B_+ = e^{+U} \tag{3a}$$

$$B_- = e^{-U} \tag{3b}$$

Clearly both B_+ and B_- are causal and they are inverses of one another. Thus, by the definition of minimum phase (see *Fundamentals of Geophysical Data Processing*) both B_+ and B_- are minimum phase.

Now we set out to establish the converse theorem, namely Theorem 3, that the logarithm of a minimum phase filter is causal. Take the logarithm of (2) and form the Z-derivative

$$U = \ln B \tag{4a}$$

$$\frac{dU}{dZ} = u_1 + 2u_2 Z + 3u_3 Z^2 + \ldots \tag{4b}$$

$$\frac{dU}{dZ} = \frac{1}{B}\frac{dB}{dZ} \tag{4c}$$

Since we assume B is minimum phase it means that both 1/B and dB/dZ on the right of (4c) are causal . Since the product of two causals is causal, we have dU/dZ causal. But clearly dU/dZ could not be causal unless U is causal.

On to Theorem 4, which says that any real power of a minimum phase function is minimum phase. Consider

$$B' = B^r = \left(e^{\ln B}\right)^r = e^{r \ln B} \tag{5}$$

Since B is assumed minimum phase, ln B by Theorem 3 will be causal. Scaling by a real constant r does not change causality. Exponentiating shows, by Theorem 2, that B is minimum phase.

Finally we will prove Theorem 5, that an impedance function can be raised to any fractional power $-1 \leq \rho \leq +1$ and the result is still an impedance function. In FGDP we find that an impedance function is defined as a causal, minimum phase function with the additional property that the real part of its Fourier transform is positive. This means that the phase angle ϕ lies in the range $-\pi/2 < \phi < +\pi/2$. Raising the impedance function to the ρ power will compress the range to $-\pi\rho/2 < \phi < \pi\rho/2$. This will keep its real part positive. Theorem 4 states that *any* power of a minimum phase function is causal, which is a lot more than we need to be certain that a *fractional* power of an impedance function will be causal.

Causal integration

Causal integration is conveniently represented in the discrete time domain by the bi-linear transform. It is shown in "Impedance, Reflectance and Transference Functions" (also in this report) that the filter that does this, namely

$$R(Z) = \frac{1}{-i\hat{\omega}} = \frac{\Delta t}{2}\frac{1+\rho Z}{1-\rho Z} \quad \text{where} \quad -1 \ll \rho < 1 \tag{6}$$

is an impedance function (causal, real part of Fourier transform is positive). In this expression Z is the unit delay operator $e^{i\omega\Delta t}$ and ρ

is a positive constant infinitesimally less than +1. At low frequencies compared to the Nyquist ($\omega\Delta t$ small) we find that $\hat{\omega}$ tends to ω.

Extrapolation

The general form for stable extrapolation problems seems to be

$$\frac{dP}{dz} = -RP \tag{7}$$

where convergence is assured by the positive real part of the impedance function R. In reflection seismology there is great interest in the square root extrapolation operator

$$R = -ik_z = \frac{-i\omega}{v}\left(1 - \frac{v^2 k_x{}^2}{\omega^2}\right)^{1/2} \tag{8}$$

At the moment we are disinterested in the space or frequency dependence of velocity, so we set $v = 1$, obtaining

$$R = [(-i\omega)^2 + k^2]^{1/2} \tag{9}$$

In (9) we would like a causal representation of the differentiation operator such as either of the following:

$$-i\hat{\omega} = \begin{cases} \dfrac{2}{\Delta t}\,\dfrac{1-\rho Z}{1+\rho Z} & -1 \ll \rho < 1 \quad \text{and} \quad Z = e^{i\omega\Delta t} \\[2ex] -i\omega + \varepsilon & \varepsilon > 0 \end{cases} \tag{10a,b}$$

We intend to establish that the following operator is an impedance function

$$R = [(-i\hat{\omega})^2 + k^2]^{1/2} \tag{11}$$

First note that $(-i\hat{\omega})$ is causal by (10), which means that $(-i\hat{\omega})^2$ is also causal. Also, k^2 is a delta function at the time origin. Thus R given by (11) is causal. Next, let us look at the phase.

Figure (1) shows how the phase of (11) is constructed from its constituents.

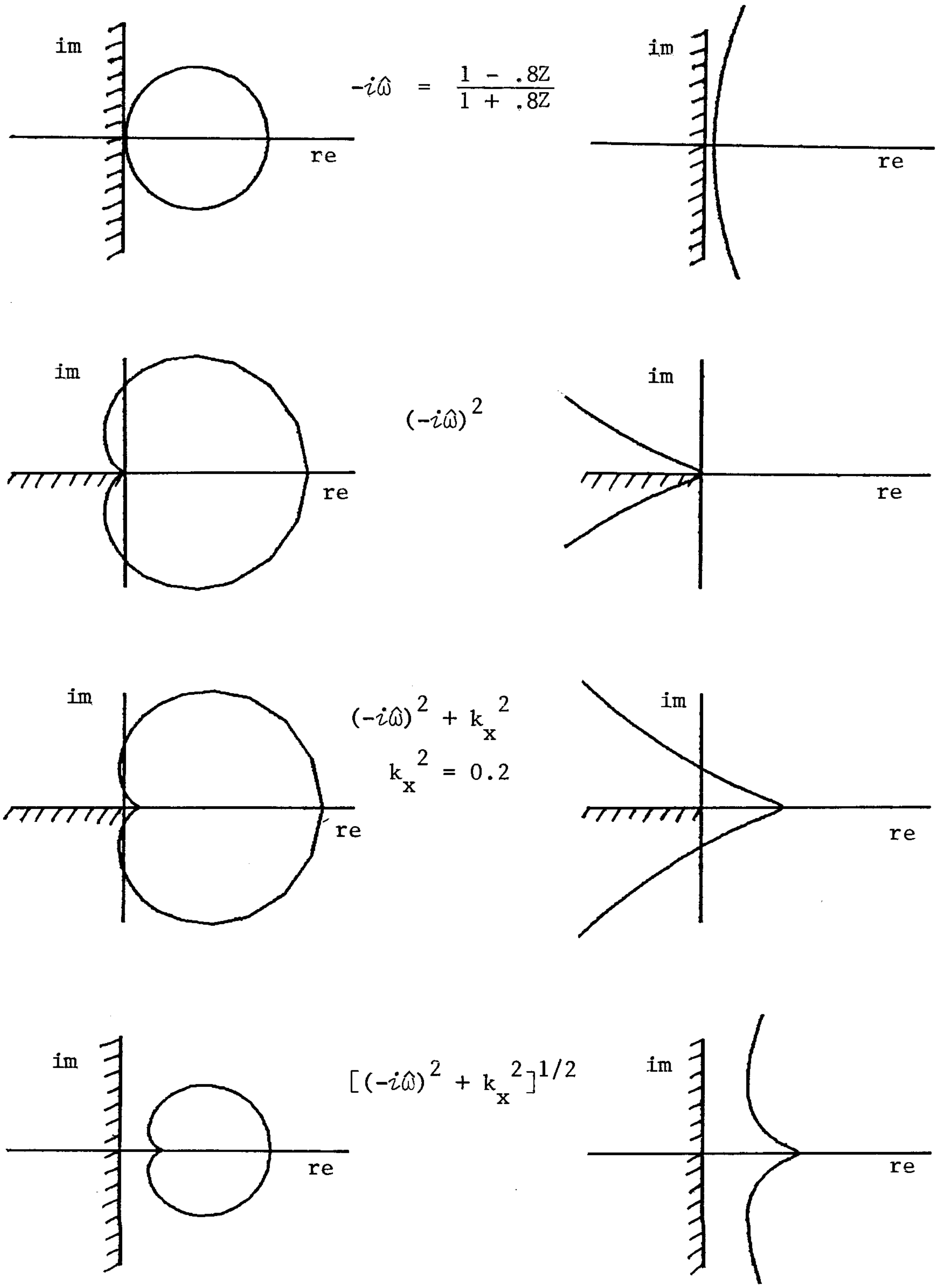

FIGURE 1.--Complex plane diagram of constituents of the extrapolation operator R as given by (11). The right column is the same as the left column blown up five times.

Now we have seen that R^2 is causal and that its phase has the "branch cut" property. That is, the phase of R has the positive real property. One of the aspects of minimum phase is that the phase does not loop around the origin. This is easily seen by inspecting

$$B = e^{U(Z)} = \exp\left(\sum_k^N U_k \cos k\,\omega + i \sum_k^N U_k \sin k\,\omega\right)$$

$$= \exp\left[r(\omega) + i\,\phi(\omega)\right]$$

Here the phase is a periodic function of ω, which means that in the plane of (Re B, Im B) the curve representing $B(\omega)$ does not enclose the origin. The branch cut forces R^2 to have this property and hence be minimum phase. Theorem 4 forces R to be causal and minimum phase. That, with the phase defined by Figure 1 proves that R, given by (9), is an impedance function. (Muir previously established that some rational approximations to R are impedance functions, but the proof does not extend to the evanescent region of the square root.)

Fractional integration and constant Q

By Equation (6) and Theorem 5 we know that fractional powers of integration and differentiation are also impedance functions. In fact, Kjartansson (1979) has advocated the fractional power as a stress-strain law for rocks. The conventional rock mechanics studies begin with a stress-strain law such as

$$\text{stress} = \text{stiffness} \times \text{strain} + \text{viscosity} \times \text{strain-rate}$$

which in transform domain is

$$\text{stress} = \left[(-i\omega)^0 \times \text{stiffness} + (-i\omega)^1 \times \text{viscosity}\right] \text{strain} \tag{12}$$

Without for the moment considering the physics of the matter, we can consider replacing the arithmetic average of the two terms by a geometric average, say

$$\text{stress} = \text{const} \times (-i\omega)^{\varepsilon}\ \text{strain} \tag{13}$$

where ε close to zero gives elastic behavior and ε close to one gives viscous behavior. The fact that $(-i\omega)^{\varepsilon}$ is an impedance function meshes nicely with the concepts that (1) stress may be determined from strain history and strain may be determined from stress history, and (2) stress times strain is work. Kjartansson (1979) points out that $(-i\omega)^{\varepsilon}$ exhibits the mathematical property called *constant Q*, so that as a stress/strain law for fitting experimental data on rocks, it is far superior to the arithmetic average. To see the constant Q property more clearly, let us express $(-i\omega)^{\varepsilon}$ in real and imaginary parts

$$
\begin{aligned}
(-i\omega)^{\varepsilon} &= |\omega|^{\varepsilon} \left[e^{-i\,\pi\,\mathrm{sgn}(\omega)/2}\right]^{\varepsilon} \\
&= |\omega|^{\varepsilon}\left\{\cos\left[\frac{\pi\varepsilon}{2}\,\mathrm{sgn}(\omega)\right] - i\,\sin\left[\frac{\pi\varepsilon}{2}\,\mathrm{sgn}(\omega)\right]\right\} \\
&= |\omega|^{\varepsilon}\left[\cos\left(\frac{\pi\varepsilon}{2}\right) - i\,\mathrm{sgn}(\omega)\,\sin\left(\frac{\pi\varepsilon}{2}\right)\right] \qquad (14)
\end{aligned}
$$

The constant Q property follows from the constant ratio between real and imaginary parts of this function. Unfortunately, we have been unable to find a closed form representation for $(-i\omega)^{\varepsilon}$ in the discrete time domain. Kjartansson (1979) gives the form in the continuum as

$$
\begin{aligned}
\mathrm{IFT}\ (-i\omega)^{\varepsilon} &= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\, t^{-1-\varepsilon} \qquad && t > 0 \qquad (15) \\
&= 0 && t < 0
\end{aligned}
$$

Although ε is permitted to range from -1 to $+1$, singularities at $t = 0$ may need to be considered separately.

The log integration operator is one side of the Hilbert Transform.

Since the causal integral operator (6) is an impedance function, by Theorem 2 it should have a causal logarithm. Defining its logarithm as U we have

$$
U(Z) = \ln \frac{1}{-i\hat{\omega}} = \ln \frac{\Delta t}{2}\,\frac{1+\rho Z}{1-\rho Z} \qquad (16)
$$

To obtain a time domain representation of U we proceed as suggested by Equation (4) and take the Z-derivative of any causal Z-transform $U(Z)$

$$\frac{dU}{dZ} = u_1 + 2u_2Z + 3u_3Z^2 + 4u_4Z^3 + \ldots \tag{17}$$

Applying d/dZ to the righthand side of (16) we get

$$\begin{aligned} \frac{dU}{dZ} &= \frac{d}{dZ}\left[\ln(\Delta t/2) + \ln(1 + \rho z) - \ln(1 - \rho z)\right] \\ &= \frac{\rho}{1 + \rho Z} + \frac{\rho}{1 - \rho Z} \\ &= 2\rho[1 + (\rho Z)^2 + (\rho Z)^4 + \ldots] \end{aligned} \tag{18}$$

Take the limit $\varepsilon \to 0$ where $\rho = 1 - \varepsilon$ and identify coefficients of like powers of Z in (17) and (18). Also substitute $Z = 0$ in (16) to find u_0. We have

$$u_k = \begin{cases} 0 \quad \text{for} \quad k \quad \text{negative} \\ \ln(\Delta t/2) \quad \text{for} \quad k = 0 \\ 2/k \quad \text{for} \quad k = 1,3,5,7,\ldots \\ 0 \quad \text{for} \quad k = 2,4,6,8,\ldots \end{cases} \tag{19}$$

What we see is that in the time domain the function $\ln[1/(-i\hat{\omega})]$ is causal and drops off as inverse time. This is just like one side of the Hilbert Transform including the discrete domain representation as inverse odd integers. In the frequency domain we have

$$\begin{aligned} \ln\left(\frac{1}{-i\omega}\right) &= -\ln(-i\omega) = -\left[\ln|\omega| - i\,\frac{\pi}{2}\,\text{sgn}(\omega)\right] \\ &= -\ln|\omega| + i\,\frac{\pi}{2}\,\text{sgn}(\omega) \end{aligned} \tag{20}$$

Adding (19) to the negative of its time reverse yields the Hilbert kernel $2/k$ for k odd. The corresponding operation on (20) naturally gives the imaginary sgn function.

$$\ln\left(\frac{1}{-i\omega}\right) - \ln\left(\frac{1}{i\omega}\right) = i\,\pi\,\mathrm{sgn}(\omega) \tag{21}$$

The Hilbert kernel is an asymmetric time function with 90-degree phase shift and no color change. The log integral is causal with slight color change and phase shift about 90 degrees in the vicinity of $|\omega| = 1$. [Do not be confused by the differing scale factor of 2 between (20) and (21). When $|\omega| = 1$, both are imaginary and odd, so that both have the same 90^{o} phase shift.]

Reflection from Q contrast

Reflections arise at an interface of impedance with a well-known reflection strength

$$C = \frac{R_2 - R_1}{R_2 + R_1} \tag{22}$$

We often think of the impedance as the velocity density product, but at non-vertical incidence the product is divided by the angle cosine of the ray. We know that $(-i\omega)^{\varepsilon}$ is also an impedance function, and we may suspect that it too could be inserted into (22) as, say R_2 with say $R_1 = 1$. This gives

$$C = \frac{(-i\omega)^{\varepsilon} - 1}{(-i\omega)^{\varepsilon} + 1} \tag{23}$$

Kjartansson (see p. , this report) has shown that this will describe the physics of a wave reflected in a medium of one constant Q value from another medium. Equation (23) is also the first term in an expansion for logarithm, and as ε tends to zero the expansion is dominated by the first term. Thus, the reflected wave takes the form

$$C = \frac{\varepsilon}{2} \log \left(\frac{1}{-i\omega} \right) \qquad (24)$$

which is expressed in the time domain by Equation (19).

REFERENCE

KJARTANSSON, E., "Constant Q -- wave propagation and attenuation," *J. Geophys. Res.*, in press 1979.