BULLET-PROOFING THE CODE FOR THE 45-DEGREE EQUATION

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Abstract

A frequency domain finite-differencing algorithm can be bullet-proofed (made unconditionally stable) by changing several lines of code in existing routines.

Introduction

Stability is one of the most desirable features that a computer algorithm can have. Other desirable features are simplicity, ease of programming, storage, and low cost. Fortunately, bullet-proofing the 45-degree procedure does not add to the complexity or cost of the program. It only requires some careful analysis.

In the migration of upcoming wavefields, we are led to consider equations of the form

$$\frac{d}{dz} U = -RU \qquad (1)$$

where R is an impedance function approximating the square root extrapolator (see "Powers of Causal Operators" by Claerbout and Kjartansson, this report). Here we specialize to the 45-degree approximation to R. We will discretize (1) with respect to x and z while preserving the stability properties of the differential equation.

How to code a bullet-proof migration

The equation that we wish to discretize is similar to a one-way wave equation described in Godfrey and Claerbout ("Stable Extrapolation," this report). Our Equation (2) controls the downward-continuation of upcoming waves (migration). It will be shown in the next section that it ensures depth invariance of the quadratic energy flux U^*VU when V is independent of z:

$$\frac{d}{dz} U = -\left\{(i\omega + \varepsilon_0)(\Lambda - S) + V^{-1}\left[\frac{VD_xD_x^tV}{2(i\omega + \varepsilon_1)I + \dfrac{VD_xD_x^tV}{2(i\omega + \varepsilon_2)}}\right]\right\} U \qquad (2)$$

with

$$D_xD_x^t = \frac{1}{\Delta x^2}\frac{T}{I - \gamma T}$$

$+\varepsilon$ for downward-continuation

$-\varepsilon$ for upward-continuation

$+\omega$ for upcoming waves

$-\omega$ for downgoing waves

$$\frac{1}{-i\omega + \varepsilon} = \text{causal integration}$$

Equation (2) is an equation for vectors U with components at discrete intervals along the x-axis. V is a real, diagonal matrix whose entries are the acoustic velocities at the locations where $U(z)$ is defined. Λ and S are also real, diagonal matrices, where $\Lambda = V^{-1}$ and S relate to splitting and retarding (see D. Brown, "Splitting and Separation of Differential Equations...," SEP-15, pp. 214-32). The real, symmetric, tridiagonal matrix T has (-1,2,-1) on its diagonal and 2's at the corners. Dip filtering is accomplished with the scalars ε_0, ε_1, and ε_2. They also keep track of causality for use in programming up the time-domain equivalent of Equation (2).

With these definitions, we can discretize Equation (2) so that it is in the form

$$(D_3 + TD_4)U' = (D_1 + TD_2)\ U(z) \tag{3}$$

where D_1, D_2, D_3, and D_4 are all complex, diagonal matrices. In the absence of splitting, $U' = U(z + \Delta z)$. The D's in Equation (3) are defined by

$$D_1(\omega,\Delta z) = (i\omega + \varepsilon_1)\left[I - (i\omega + \varepsilon_0)\frac{\Delta z}{2}(\Lambda - S)\right] \tag{4a}$$

$$D_2(\omega,\Delta z) = \left[\frac{1}{4(i\omega + \varepsilon_2)\Delta x^2} V^2 - \gamma(i\omega + \varepsilon_1)I\right] \left[I - \frac{\Delta z}{2}(i\omega + \varepsilon_0)(\Lambda - S)\right] - \frac{\Delta z}{4\Delta x^2} V \tag{4b}$$

$$D_3(\omega,\Delta z) = D_1(\omega,-\Delta z) \tag{4c}$$

$$D_4(\omega,\Delta z) = D_2(\omega,-\Delta z) \tag{4d}$$

A proof of the validity of (4) can be found in Appendix A.

Equation (4) can be altered to handle downgoing as well as upcoming waves and to extrapolate upward as well as downward.

Downward-continuation of upcoming waves (migration):

$$[D_3(\omega) + TD_4(\omega)]\ U(z + \Delta z) = [D_1(\omega) + TD_2(\omega)]\ U(z) \tag{5}$$

Upward-continuation of upcoming waves (diffraction or modeling):

$$[D_3(-\omega) + TD_4(-\omega)]\ U(z - \Delta z) = [D_1(-\omega) + TD_2(-\omega)]\ U(z) \tag{6}$$

Upward-continuation of downgoing waves (migration?):

$$[D_3(\omega) + TD_4(\omega)]\ D(z - \Delta z) \;=\; [D_1(\omega) + TD_2(\omega)]\ D(z) \tag{7}$$

Downward-continuation of downgoing waves (diffraction):

$$[D_3(-\omega) + TD_4(-\omega)]\ D(z + \Delta z) \;=\; [D_1(-\omega) + TD_2(-\omega)]\ D(z) \tag{8}$$

Let us verify that these signs are consistent with (2). Going from (5) to (8), the sign of ω changes because of the switch from up- to downgoing waves. Going from (5) to (6) means a switch from downward- to upward-continuation and therefore changes of the signs of the ε_n in (2). But (6) projects from z to $z - \Delta z$, which is like a change of sign on the z-axis in (2). The combined effect is to change the sign of ω in going from (5) to (6). In (7) we have changed all three signs (z,ε,ω) so the result is identical to (5).

Basic physics and a more accurate equation

The methods of Godfrey, Muir and Claerbout ("Stable Extrapolation," this report) show that the solution q of

$$\frac{d}{dz}\, q \;=\; -\left\{ i\omega(\Lambda - S) + V^{-1/2} \left[\frac{VD_x D_x^t V}{2i\omega I + \dfrac{VD_x D_x^t V}{2i\omega}} \right] V^{-1/2} \right\} q \tag{9}$$

has the property that q^*q is independent of z. Asserting that the energy flux across a datum at any particular z-level equals that at any other z-level, we are led to identify q^*q with this vertical component of energy flux. The appearance of $V^{-1/2}$ in two places in (9) was considered to be a slight programming inconvenience. A change of variables to $U = V^{-1/2}q$ along with the approximation that $V \neq V(z)$

reduces the exact energy conserving form (9) to the form of (2), which we judged to be slightly more convenient. By substitution of $q = V^{1/2}U$ the conservation of q^*q is equivalent to the conservation of u^*vu, a fact that justifies our assertion that (2) forms the basis of a stable algorithm.

According to *Fundamentals of Geophysical Data Processing*, the acoustic energy flow across a datum is the admittance times the pressure P squared, say

$$q^*q \quad = \quad U^*VU \quad = \quad P^* \frac{\cos\theta}{pv} P \tag{10}$$

which enables us to relate energy flux variables q and u to acoustic variables, pressure P, ray angle θ, velocity v, and density p. For those who prefer the accuracy of (9) to the convenience of (2), the revised coding instructions are:

$$(D_3 + TD_4)q' \quad = \quad (D_1 + TD_2)\, q(z)$$

$$D_1(\omega,\Delta z) \quad = \quad (i\omega + \varepsilon_1)\, \Lambda^{1/2}\, [I - (i\omega + \varepsilon_0)\, \frac{\Delta z}{2}\, (\Lambda - S)]$$

$$D_2(\omega,\Delta z) \quad = \quad \left[\frac{1}{4(i\omega + \Delta\varepsilon_2)\Delta x^2}\, V^2 - \gamma(i\omega + \varepsilon_1)I\right] \Lambda^{1/2} \left[I - \frac{\Delta z}{2}\, (i\omega + \varepsilon_0)(\Lambda - S)\right] - \frac{\Delta z}{4\Delta x^2}\, V^{1/2} \tag{11}$$

$$D_3(\omega,\Delta z) \quad = \quad D_1(\omega,-\Delta z)$$

$$D_4(\omega,\Delta z) \quad = \quad D_2(\omega,-\Delta z)$$

The demonstration of Equation (11) is in Appendix B.

APPENDIX A

We want to factor the Crank-Nicolson approximation to Equation (2) so that it is in the form of Equation (3):

$$(D_3 + TD_4)U' = (D_1 + TD_2)\ U(z)$$

Let R denote the quantity in braces in Equation (2), and let r_1, r_2, and r_3 stand for $i\omega + \varepsilon_0$, $i\omega + \varepsilon_1$, and $i\omega + \varepsilon_2$, respectively. In this case, the Crank-Nicolson approximation to Equation (2) is

$$U' = (I + \frac{\Delta z}{2} R)^{-1} (I - \frac{\Delta z}{2} R)\ U(z)$$

$$R = r_1(\Lambda - S) - 2\Lambda\left(r_2 I + \frac{VD_x D_x^t V}{4r_3}\right)^{-1} VD_x D_x^t V$$

If we let

$$a = \frac{\Delta z}{2}$$

$$A^{-1} = (r_2 I + \frac{1}{4r_3} VD_x D_x^t V)^{-1}$$

$$D = I - \frac{r_1 \Delta z}{2} (\Lambda - S)$$

then

$$\begin{aligned} I - aR &= D - \frac{a}{2} V^{-1}A^{-1}VD_x D_x^t V \\ &= V^{-1}A^{-1}(AVD - \frac{a}{2} VD_x D_x^t V) \\ &= V^{-1}A^{-1}[(r_2 I + \frac{1}{4r_3} VD_x D_x^t V)VD - \frac{a}{2} VD_x D_x^t V] \\ &= V^{-1}A^{-1}V(I - \gamma T)^{-1}\{[r_2(I - \gamma T)V^{-1} + \frac{1}{4r_3 \Delta x^2} TV]\ VD \\ &\quad - \frac{a}{2\Delta x^2} TV\} \end{aligned}$$

The quantity $V^{-1}A^{-1}V(I - \gamma T)^{-1}$ is independent of $a = \frac{\Delta z}{2}$ so

$$(I + aR)^{-1}(I - aR) = \{[r_2(I - \gamma T)\Lambda + \frac{1}{4r_3\Delta x^2} TV]VD + \frac{a}{2\Delta x^2} TV\}^{-1}$$

$$\{[r_2(I - \gamma T)\Lambda + \frac{1}{4r_3\Delta x^2} TV]VD - \frac{a}{2\Delta x^2} TV\}$$

$$= (D_3 + TD_4)^{-1}(D_1 + TD_2)$$

APPENDIX B

Using the notation of Appendix A, we get a Crank-Nicolson factor from Equation (9):

$$
\begin{aligned}
I - aR &= D - \frac{a}{2}\Lambda^{1/2}A^{-1}VD_xD_x^tV\Lambda^{1/2} \\
&= \Lambda^{1/2}A^{-1}(AV^{1/2}D - \frac{a}{2}VD_xD_x^tV^{1/2}) \\
&= \Lambda^{1/2}A^{-1}[(r_2I + \frac{1}{4r_3}VD_xD_x^tV)V^{1/2}D - \frac{a}{2}VD_xD_x^tV^{1/2}] \\
&= \Lambda^{1/2}A^{-1}V(I - \gamma T)^{-1}\{[r_2(I - \gamma T)\Lambda + \frac{1}{4r_3\Delta x^2}TV]V^{1/2}D \\
&\qquad - \frac{a}{2\Delta x^2}TV^{1/2}\}
\end{aligned}
$$

from which:

$$
\begin{aligned}
(I + aR)^{-1}(I - aR) &= \{[r_2(I - \gamma T)\Lambda + \frac{1}{4r_3\Delta x^2}TV]V^{1/2}D + \frac{a}{2\Delta x^2}TV^{1/2}\}^{-1} \\
&\qquad \{[r_2(I - \gamma T)\Lambda + \frac{1}{4r_3\Delta x^2}TV]V^{1/2}D - \frac{a}{2\Delta x^2}TV^{1/2}\}
\end{aligned}
$$