# STABLE EXTRAPOLATION

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Stable extrapolation can be assured by preserving certain symmetries. Given the differential equation

$$\frac{dq}{dz} = -Rq \tag{1}$$

and its Crank-Nicolson approximation

$$\frac{q_{n+1} - q_n}{\Delta z} = -\frac{R}{2}(q_{n+1} + q_n) \tag{2}$$

it will be shown that stability is assured in both cases, provided that $R + R^*$ is a positive definite (actually, semi-definite) matrix. In a paper entitled "Impedance, Reflectance, and Transference Functions" (see this report) this subject is developed with the operator R being a scalar Z-transform. There the matter of *causality* receives much attention, whereas here we focus more on the *matrix* character of R. It seems that the present work, along with "Impedance....." provides a theory for causal matrix extrapolation that is useful in time domain migration. Our purpose for this theoretical work is to enable us to write a "bullet proof" program for migrating seismic data in the presence of lateral velocity variation. As a final example we will see that the familiar "45-degree" extrapolation equation can be put in the required form.

## *Stability of the differential equation*

Let $q^*$ denote the Hermitian conjugate of q. For Equation (1) to be stable, we require the energy $q^*q$ to be either constant or decaying as we extrapolate in depth $z$:

$$\frac{d}{dz}(q^*q) \leq 0$$

$$q^*q_z + q_z^*q \leq 0 \tag{3}$$

Substituting Equation (1) into (3) gives

$$q^*Rq + q^*R^*q \geq 0$$

$$q^*(R + R^*)q \geq 0 \tag{4}$$

Equation (4) shows that $R + R^*$ must be positive semi-definite for the differential equation to be stable, which is what we set out to prove.

*Stability of the difference equation*

The stability of the difference equation is shown in a similar way, but there is some extra clutter. Incorporating some ideas contained in a letter from Björn Engquist, we reduced the clutter to what is found below. First observe the identity:

$$(a^*a - b^*b) \equiv \frac{1}{2}[(a + b)^*(a - b) + (a - b)^*(a + b)] \tag{5}$$

Letting $a = q_{n+1}$ and $b = q_n$, Equation (5) becomes

$$(q_{n+1}^*q_{n+1} - q_n^*q_n) = \frac{1}{2}[(q_{n+1} + q_n)^*(q_{n+1} - q_n) + (q_{n+1} - q_n)^*(q_{n+1} + q_n)] \tag{6}$$

Now, replace the $(q_{n+1} - q_n)$ terms by using Equation (2):

$$= -\frac{\Delta z}{4}[(q_{n+1} + q_n)^*R(q_{n+1} + q_n) + (q_{n+1} + q_n)^*R^*(q_{n+1} + q_n)]$$

$$= -\frac{\Delta z}{4}\left[(q_{n+1} + q_n)^*(R + R^*)(q_{n+1} + q_n)\right] \tag{7}$$

This equation establishes the final result: If $R + R^*$ is a positive definite matrix, then $q^*_{n+1}q_{n+1}$ is less than $q^*_nq_n$.

*Application to 45-degree wavefield extrapolation*

The scalar wave equation for extrapolation of a downgoing wavefield is

$$\frac{dq}{dz} = ik_z q = -Rq \tag{8}$$

where the $R$ operator takes the usual form

$$R = -ik_z = \frac{-i\omega}{v}\left(1 - \frac{v^2 k_x{}^2}{\omega^2}\right)^{1/2} \tag{9}$$

Our plan now is to approximate the square root by the usual series expansion and then identify $ik_x$ with $\partial_x$ to obtain a space domain equation. The main effort stems from our refusal to make the usual assumption that $v(x,z)$ is independent of $x$. Since $\partial_x vq$ differs from $v\partial_x q$ the space representation does not seem to be unique, and we may wonder how the variable $q$ relates to physical wave variables like pressure and displacement. Since (9) is purely imaginary, we will have depth invariance of the quadratic $q^*q$, which we can interpret as the downward energy flux across the datum at depth $z$. Our main effort will be to assure that $q^*q$ does indeed remain depth-invariant when we incorporate $v(x,z) \neq$ const. The task of determining the relation between the energy flux variable $q$ and the physical variables will be left to the user.

First of all, we will need to consider the representation of $v^2 k_x^2$ in the space domain. Thinking of the x-derivative operator $\partial_x$ as a large bidiagonal matrix with $(1,-1)/\Delta x$ along the diagonal and $V(x)$ as a diagonal matrix, we are attracted to expressions like $(V\partial_x)^T(V\partial_x)$ or $(V\partial_x)(V\partial_x)^T$ because they are symmetric positive semi-definite matrices. In simplest form, such numerical representations are tridiagonal matrices that we will abbreviate as

$$T = \begin{cases} (V\partial_x)(V\partial_x)^T \\ \\ (V\partial_x)^T(V\partial_x) \end{cases} \qquad \text{(10a,b)}$$

At a later time accuracy or some other consideration could make the choice in (10). Even other expressions can be used provided they are real, symmetric,and positive definite.

The usual constant velocity 45-degree expansion of (9) from "Impedance, Reflectance and Transference Functions" is

$$R = \frac{1}{v}\left( -i\omega + \frac{v^2 k_x^2}{-i\omega 2 + \frac{v^2 k_x^2}{-i\omega 2}} \right) \qquad \text{(11)}$$

This scalar $R$ always has a positive real part because $-i\omega$ is always replaced by the causal operator $(-i\omega + \varepsilon)$ where $\varepsilon$ is positive and the expression follows Muir's rules for compounding impedances. In going to the x-domain we note that $(ik_x)^2 = -\partial_{xx}$ and $(\partial_x)^T = -\partial_x$. So the positive scalar $v^2 k_x^2$ corresponds to the positive eigenvalues of (10). The expression of $R$ in the space domain will now be given as

$$M = -i\omega I + \frac{T}{-i\omega 2I + \frac{T}{-i\omega 2}} \qquad \text{(12a)}$$

$$R = V^{-1/2}\, M\, V^{-1/2} \qquad \text{(12b)}$$

Use of the division sign in (12) is justifiable because the matrix T commutes with the identity matrix I. (A hazard in this kind of effort is that T does not commute with the diagonal matrix V.) The matrix M has the properties required of R since a basic matrix theorem says that the eigenvalues of a polynomial of a real symmetric matrix are the polynomials of the eigenvalues. In other words, replacing T in (12) by one of its eigenvalues produces a complex M whose real part is positive so that M* + M is positive as required. What we need to show is that the following matrix is positive definite:

$$R + R^* = V^{-1/2}(M + M^*)V^{-1/2} \tag{13}$$

A matrix A is positive definite if for arbitrary x, the scalar x*Ax is positive. The diagonal matrix $V^{-1/2}$ can certainly be absorbed into x and x will still be arbitrary, so the proof is complete.

*Extensions*

The following notes are added without proof. They provide alternative avenues of approach.

1. Let A = (I - R)/(I + R). Then (2) will be stable if (I - A*A) is positive definite.

2. In programming it is a nuisance to put $V^{-1/2}$ on each side of the matrix M. Actually you can put $V^{-1}$ on either side. In general, some other quadratic form such as q*Uq where U is strictly positive definite will be decreasing if

   $$R^*U + UR$$

   is positive definite.


"Eureka exclamation point . . ."