GEOMETRICAL OPTICS
AND WAVE THEORY OF CONSTANT OFFSET SECTIONS
IN LAYERED MEDIA

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Abstract

The theory of constant offset sections discussed in this paper allows them to be mapped to a fixed offset and compared in order to provide a method of velocity analysis. The mapping to *zero* offset might provide an alternative processing procedure to NMO and stack. Regretably, the theory does not provide exact solutions in closed form. Two alternative sets of approximate expressions for the frequency/wavenumber migration of constant offset sections are derived for layered media. These are the low offset approximation valid for all angles of propagation and the 15° approximation valid for all offsets. The angular amplitude variation for a migration operator is also derived from the double square root equation.

Introduction

The objective of this paper is to promote a better understanding of migration principles as applied to constant offset sections. We will interrelate geometrical optics with wavefield concepts and employ the double square root equation (Clayton, SEP-14; Claerbout, SEP-15) to define direct mappings from the gathers of the seismic field experiment to zero-offset sections, and hence, ultimately to the migrated seismic section. These mappings are obtained by means of the *principle of stationary phase,* which allows us to "ray trace" down to a reflector and then back up again to the surface. The velocity appearing in our equations is the true velocity v and has to be replaced by the half-velocity

$c = v/2$ in order to relate the mappings to the equations obtained under the exploding reflector concept.

En route we develop a "smear-stack" operator for mapping constant offset gathers onto zero offset. The idea is to illustrate how geometrical optics can be used to define the key components of corresponding wavefield operators rather than to propose a practical algorithm.

Mapping to zero offset: geometrical optics argument

We start with a constant offset gather in the (y,t)-plane where y is the midpoint and t the two-way traveltime. This plane is now seeded with a single impulse at $y = y_h$, $t = t_h$ (Figure 1), and the equation of the corresponding reflector in the (x',τ)-plane is found, where τ is the one-way vertical traveltime. The equation is clearly $t_s + t_g = t_h$ or just the classic ellipse, which for $y_h = 0$ becomes:

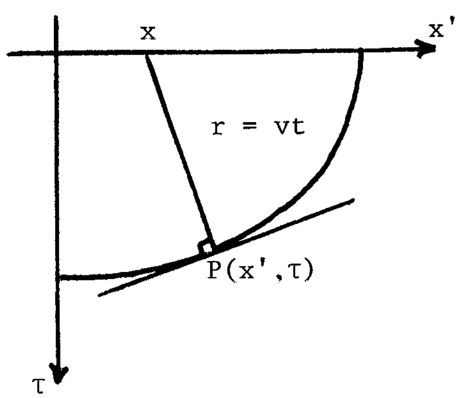
$$\frac{(x')^2}{a^2} + \frac{\tau^2}{b^2} = 1$$

with $\quad a = \frac{1}{2} v t_h$

and $\quad b = \left(\frac{t_h^2}{4} - \frac{h^2}{v^2} \right)^{1/2} = \frac{t_0}{2}$

where $h = \frac{1}{2} f$ is the half-offset, and t_0 is the two-way zero-offset vertical traveltime.

The next stage is to simulate the zero-offset experiment for this reflector. Each point $P(x',\tau)$ on the elliptical reflector maps to an image point (x,t) in the zero-offset section, where $(x,0)$ is the surface point of intersection with the normal to the ellipse at P, and $t = 2r/v$ is the two-way traveltime. Using the parametric form



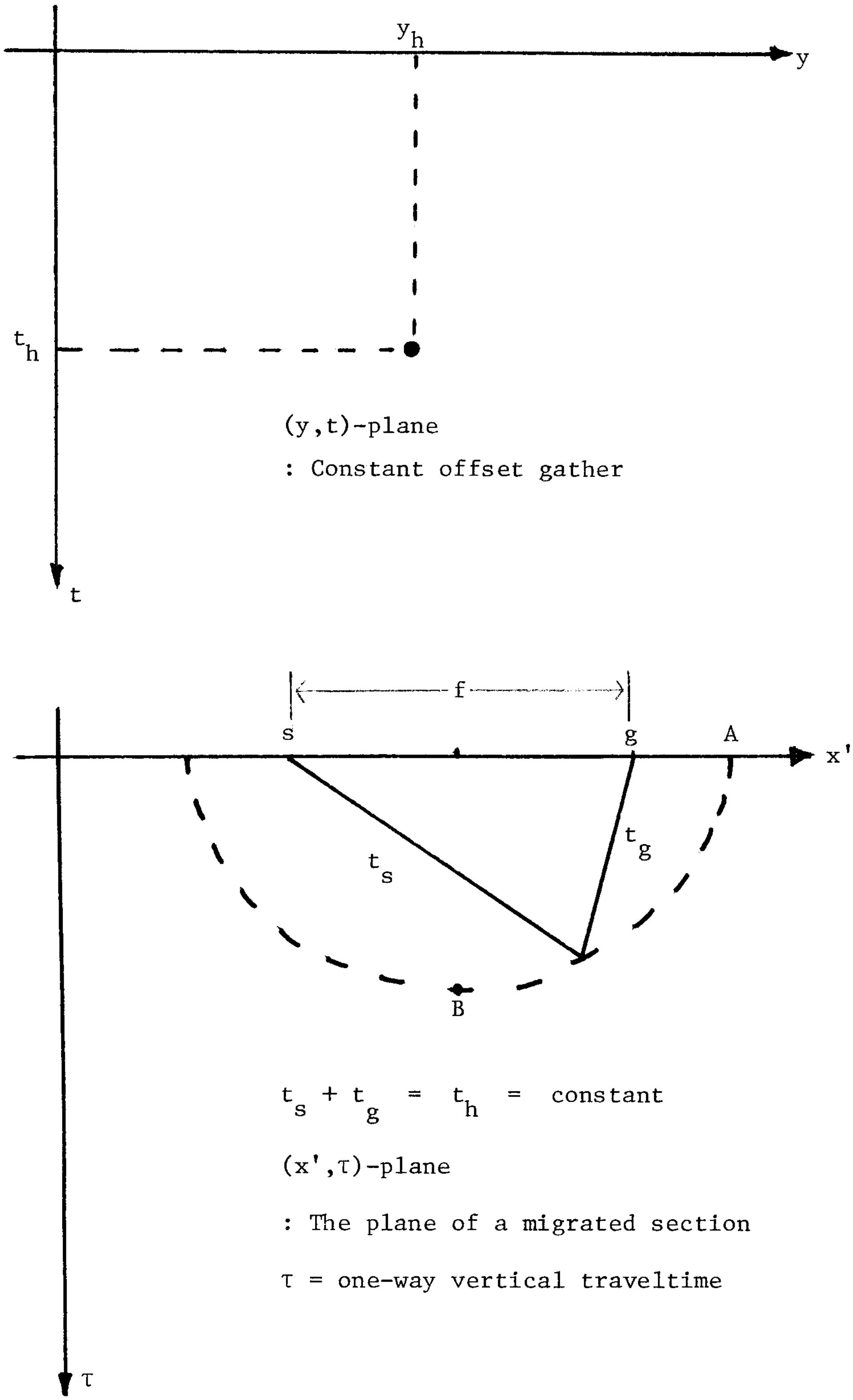


FIGURE 1.--Mapping from (y,t) to (x',τ).

of the ellipse ($x = a\cos\phi$, $\tau = b\sin\phi$) we obtain

$$x = \left(a - \frac{v^2b^2}{a}\right)\cos\phi$$

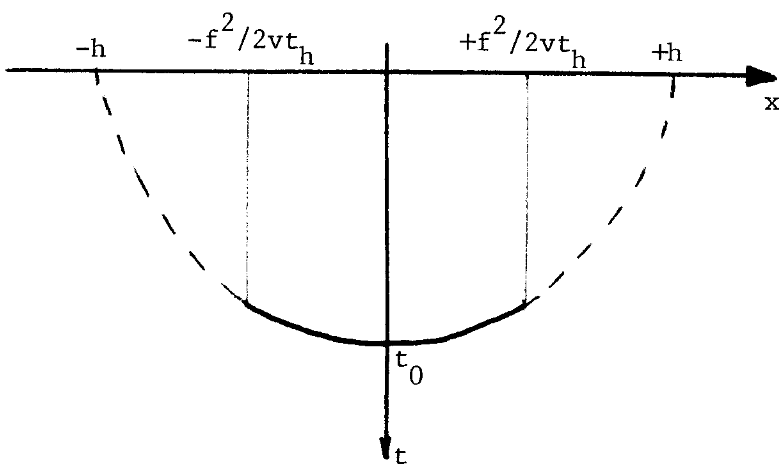
and

$$t^2 = \frac{4r^2}{v^2} = 4b^2\sin^2\phi + \frac{4v^2b^4}{a^2}\cos^2\phi$$

On substituting for a and b we achieve the equation of the required smile. This defines the curve along which energy at any point (y_h, t_h) on the non-zero offset section is to be "smeared" when mapped to zero offset:

$$x = \frac{f^2}{2vt}\cos\phi$$

$$t = t_0\left(\sin^2\phi + \frac{t_0^2}{t_h^2}\cos^2\phi\right)^{1/2}$$



This smile is seen to be laterally bounded according to $|x| \leq x_m = f^2/2vt_h$ ($\cos\phi = \pm 1$) and to bottom at t_0, and is itself part of a larger ellipse with velocity-independent semi-axes h, t_0. In the section on layered media it is shown that the axes of the corresponding ellipse are approximated by Mh, t_0, where

$$M^2 = \frac{t_0 \int_0^{t_0} v^4 \, dt}{\left(\int_0^{t_0} v^2 \, dt \right)^2}$$

The mapping defined by the smile may now be written in terms of the smear-stack operator S:

$$P_0(x,t) = \iint P_h(y,t_h) \cdot S(x-y,t-t_h) \, dy \, dt_h$$

where

$$S(x,t) = L(x) \, \delta\left[t - t_0 \left(1 - \frac{x^2}{h^2} \right)^{1/2} \right] \qquad |x| < x_m, \quad t > 0$$

$$= 0 \quad \text{otherwise}$$

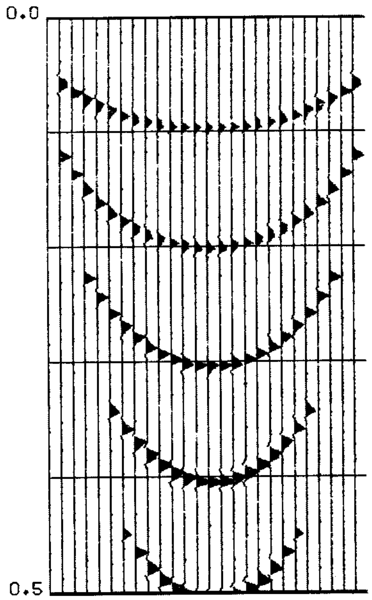
L(x) defines the amplitude distribution along the smile. Although wave theory could be used for its definition, we will simply take the amplitude to be proportional to the curvature of the elliptical reflector; in which case:

$$L(x) \propto \frac{vt_0}{2h^2} (1 - Ax^2)$$

with

$$A = \frac{3}{2h^2} \left(\frac{v^2 t_0^{\,2}}{4h^2} - 1 \right)$$

A simple algorithm was written for applying the above time/space-variant convolution via a "smear-stack" with L(x) normalized to unit dc. Note that under the above definition A is not necessarily positive. For either shallow events or high offsets it will become negative, implying an increasing rather than a decreasing amplitude toward the extrema. This may be witnessed in the top smile of Figure 2, which corresponds to an offset of 30 traces or 750 m; Figure 3 shows the smear operator reducing to a pure impulse for deeper events at a lower offset of 500 m.



velocity = 2500 m/sec

δx = 25m

grid size = 25 x 100

Fig. 2: δt = 0.005 sec

offset = 30 traces = 750 m

Fig. 3: δt = 0.016 sec

offset = 20 traces = 500 m

FIGURE 2.--Smiles in close up.

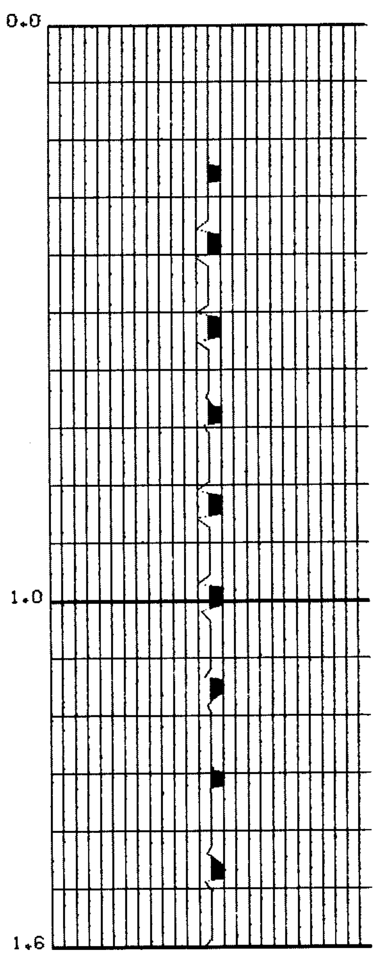


FIGURE 3a.--Constant offset section (y,t).

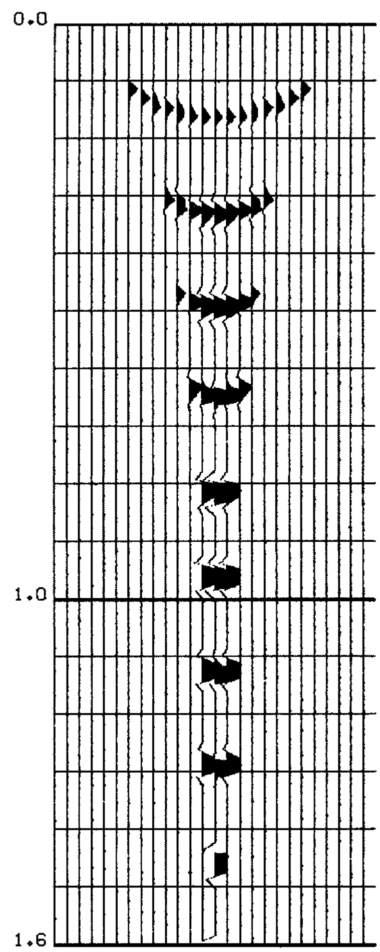
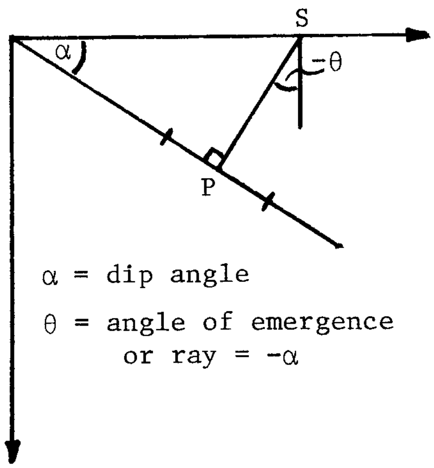


FIGURE 3b.--Mapped section (x,t_0).

From ray paths to waves

In the next section we will Fourier transform the smear operator. Here we use geometrical optics to deduce the expected form of the transform and also to derive a few results that will prove useful in later sections.

Consider a zero-offset section. The normally reflected energy from the planar element at P with dip α will have a two-way traveltime of $t = (2x/v) \sin \alpha$, where x is the coordinate of the shot/geophone position S relative to an origin at the point of emergence of the plane reflector. Differentiation of the two-way traveltime yields $dt/dx = (2/v) \sin \alpha$ and defines the reciprocal velocity at which a planar portion of the reflected wavefront intercepts the x-axis on arrival at S. We follow the SEP convention and adopt:



$$\exp[i(kx + k_z z - \omega t)]$$

for our plane wave eigenfunction, in which case,

$$F.T(\partial_t) = -i\omega \quad \text{whilst} \quad F.T(\partial_x) = ik$$

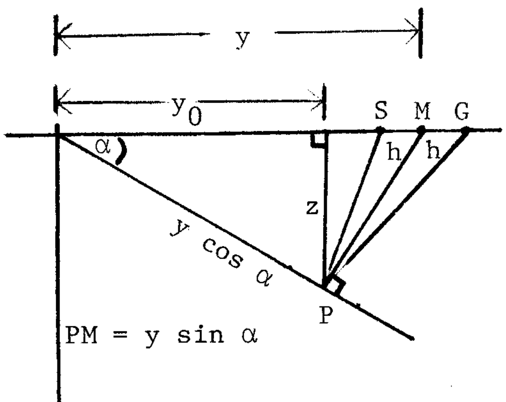
Hence, the time differential $dt/dx = (2/v)\sin \alpha$ becomes $k/\omega = -(2/v)\sin \alpha$ in the transform domain, or

$$\sin \theta = \frac{kv}{2\omega}$$

An identical result can naturally be obtained by substitution of the plane wave eigenfunction into the scalar wave equation, which, in terms of two-way traveltime t, is

$$\left(\partial_x^{\ 2} + \partial_z^{\ 2} - \frac{4}{v^2} \partial_t^{\ 2}\right) P = 0$$

We will now apply geometrical optics to a non-zero-offset section and employ the small offset, deep reflector approximation by regarding the midpoint as the recipient of the reflected ray.



From the geometry:

$$y - y_0 = y \sin^2\alpha$$

$$z = y \sin\alpha \cos\alpha$$

and the total travel path SPG is of length:

$$d = \overline{P}S + \overline{P}G$$

$$= [(y - y_0 - h)^2 + z^2]^{1/2} + [(y - y_0 + h)^2 + z^2]^{1/2}$$

As the half-offset h is assumed small, we employ the binomial expansion to obtain

$$d \approx 2y \sin\alpha + \frac{h^2 \cos^2\alpha}{y \sin\alpha}$$

$$\text{or} \qquad t = \frac{d}{v} \approx t_0 + \frac{2h^2 \cos^2\theta}{v^2 t_0}$$

which reveals the well-known $v/\cos\theta$ term, required if NMO is to be correctly applied to dipping events. Also note in passing that the vertical depth of the reflecting element is given by

$$z = \frac{vt_0}{2} \cos\theta = \frac{vt_0}{2}\left(1 - \frac{k^2 v^2}{4\omega^2}\right)^{1/2}$$

a result that will prove useful later.

We will now write

$$t = t_0 + \Delta t_N$$

where t is the observed two-way traveltime

t_0 is the (required) zero-offset traveltime

and Δt_N is the moveout operator required to map constant offset sections to zero offset

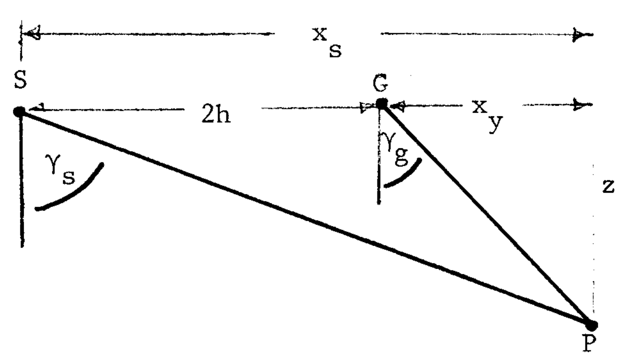
Hence,

$$\begin{aligned}\Delta t_N &\approx \frac{2h^2}{v^2 t_0}\cos^2\theta \\ &= \frac{2h^2}{v^2 t_0}\left(1 - \frac{k^2 h^2}{4\omega^2}\right) \\ &= \Delta t_n + \Delta t_d\end{aligned}$$

where Δt_n is the standard normal moveout term and Δt_d is the dip moveout. In the Fourier domain the above corresponds to a "shift" operator of the form

$$\begin{aligned}S_N &= \exp(-i\omega\Delta t_N) \\ &= \exp(-i\omega\Delta t_n)\cdot\exp(-i\omega\Delta t_d) \\ &= S_n \cdot S_d\end{aligned}$$

The angle θ used so far is the angle associated with the zero-offset two-way traveltime geometry of the idealized seismic section. We will now deal with the one-way time geometry of the constant offset section and introduce the angles γ_s, γ_g subtended at the shot and geophone by a ray path at an arbitrary diffractor point P, at depth z.



Now $\sin\gamma_s = vk_s/\omega = vp_s$, $\sin\gamma_g = vk_g/\omega = vp_g$, where k_s, k_g are the wavenumbers of the planar wavefront components emerging from the shot and incident on the geophone, respectively, whilst p_s and p_g are the corresponding Snell parameters for the ray paths. As the shot/geophone separation is 2h, we may write:

$$2h = x_s - x_g$$

$$= z(\tan \gamma_s - \tan \gamma_g)$$

$$= z\left(\frac{vp_s}{(1 - v^2 p_s^2)^{1/2}} - \frac{vp_g}{(1 - v^2 p_g^2)^{1/2}}\right)$$

We now introduce the midpoint wavenumber

$$k_y = k_s + k_g$$

and also define

$$F(k) = \frac{\frac{zvk}{\omega}}{\left(1 - \frac{v^2 k^2}{\omega^2}\right)^{1/2}}$$

in which case the previously derived ray-trace expression may be rewritten:

$$2h = F(k_s) - F(k_y - k_s)$$

This equation will provide a link between geometrical optics and wave theory in the form of the double square root equation. At zero offset the ray paths SP, GP become coincident $(k_s = k_g)$ so that $k_s = (1/2)k_y$. In this context we can regard the general offset solution as a "perturbation" of the zero-offset case with

$$k_s = \frac{1}{2} k_y - \Delta k$$

Thus, the correcting wavenumber Δk is given by

$$\Delta k = \frac{1}{2} k_y - k_s$$

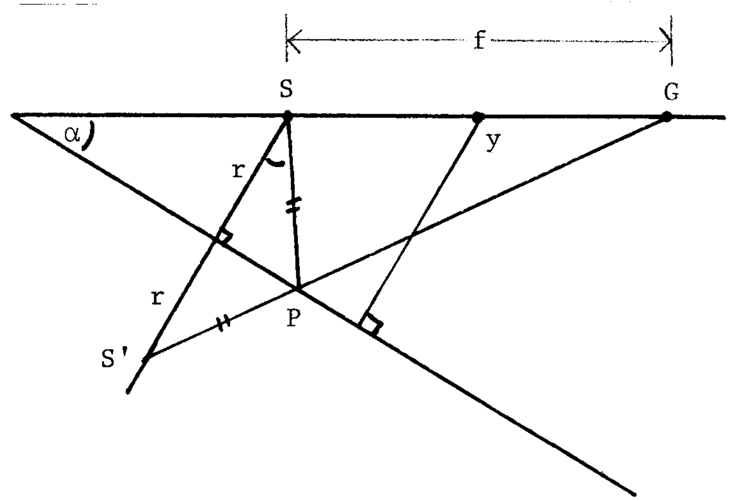
$$= \frac{1}{2} (k_g - k_s)$$

$$= \frac{1}{2} k_h$$

where k_h is the half-offset wavenumber. We may therefore rewrite the ray trace equation:

$$2h = F\left[\frac{1}{2}(k_y - k_h)\right] - F\left[\frac{1}{2}(k_y + k_h)\right]$$

Before moving on to the Fourier transform of the smear-stack operator (next section) we will look at a few examples of it in action on dipping events. Figure 4 shows a ray-traced constant offset section and Figure 5 the result of applying the smear-stack to it. The corresponding idealized zero-offset section can be seen in Figure 6.



The constant offset section was generated according to

$$t^2 = \frac{[f^2 + 4r(r + f \sin \alpha)]}{v^2}$$

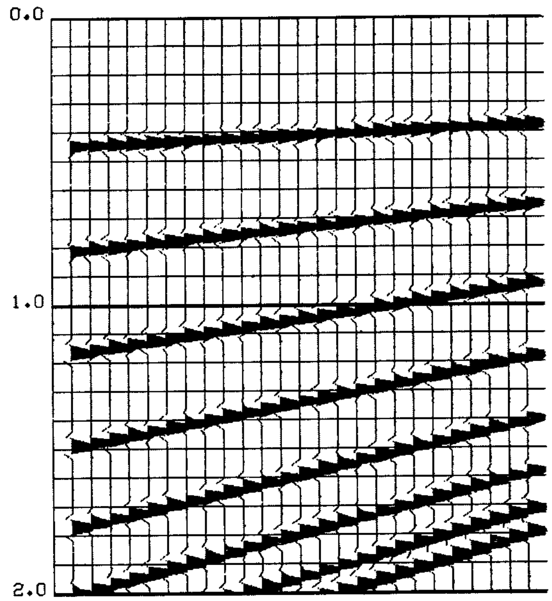
where $r = [y - (1/2\ f)]\sin \alpha$; $t_0 = (y \sin \alpha)/v$. This is the exact equation for a constant velocity medium and can be derived by applying the law of cosines to triangle SS'G.

Fourier analysis of the smear operator

Ignoring amplitudes, we can regard the smear operator as defined in terms of the δ-function:

$$\delta(x,t) = \delta\left[t - t_0\left(1 - \frac{x^2}{h^2}\right)^{1/2}\right] \qquad |x| < x_m; \quad t > 0$$

$$= 0 \qquad \text{otherwise}$$



offset = 20 traces = 500 m

velocity = 2500 m/sec

δx = 25 m;   δt = 0.02 sec

f/v = 0.2 sec

grid size = 25 x 100

FIGURE 4.--Constant offset section.

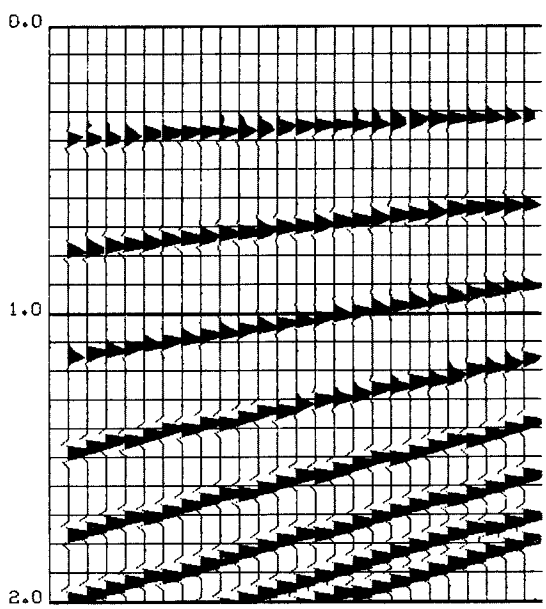


FIGURE 5.--Zero-offset mapping. Constant offset section mapped to zero offset via smear stack operator.

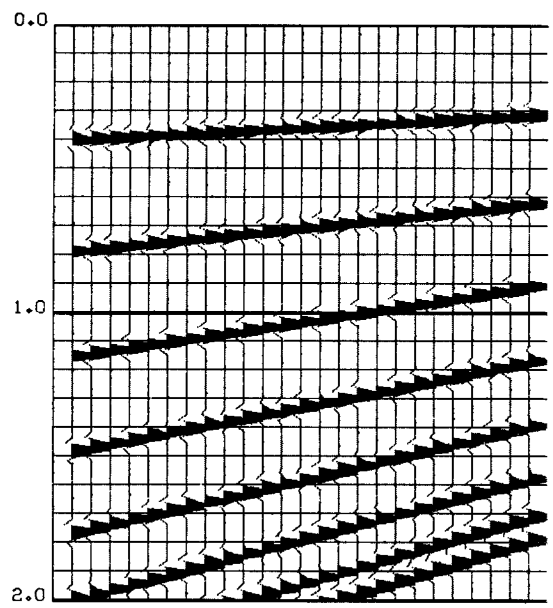


FIGURE 6.--Zero-offset section: the desired result.

We now transform the above into the (ω,k)-domain:

$$\bar{S}(\omega,k) = \int_0^{\infty} dt \int_{-x_m}^{x_m} dx\ S(x,t)\ \exp[-i(kx - \omega t)]$$

$$= \int_{-x_m}^{x_m} dx\ \exp\left\{-i\left[kx - \omega t_0\left(1 - \frac{x^2}{h^2}\right)^{1/2}\right]\right\}$$

Assume that x_m is small, so that within the range of integration,

$$\left(1 - \frac{x^2}{h^2}\right)^{1/2} \approx 1 - \frac{x^2}{2h^2}$$

in which case,

$$\bar{S}(\omega,k) = \int_{-x_m}^{x_m} dx\ \exp\left\{-i\left[kx - \omega t_0\left(1 - \frac{x^2}{2h^2}\right)\right]\right\}$$

But as x_m is assumed small, we introduce the integration factor $\exp(-Ax^2)$, where A should ideally be chosen such that the integration factor approximates to unity within the range of integration and to zero outside it. Alternatively, $\exp(-Ax^2)$ may be interpreted as an imposed amplitude variation along the smile. In either case, it allows us to extend the range of integration to infinity.

$$\bar{S}(\omega,k) = e^{i\omega t_0} \int_{-\infty}^{\infty} dx\ e^{-Ax^2} \exp\left[-i\left(kx + \frac{\omega t_0 x^2}{2h^2}\right)\right]$$

and obtain the result by reference to standard tables of integrals, from which, for $A > 0$,

$$\bar{S}(\omega,k) = e^{i\omega t_0} \frac{\pi^{1/2} h}{\left(h^4A^2 + \frac{\omega^2 t_0^{\,2}}{4}\right)^{1/4}} \cdot \exp \frac{-Ak^2}{4A^2 + \frac{\omega^2 t_0^{\,2}}{h^4}}$$

$$\cdot \exp\left[-i\left(\phi - \frac{\frac{1}{2} t_0 h^2 \omega k^2}{4h^4A^2 + \omega^2 t_0^{\,2}}\right)\right]$$

$$\text{with} \quad \phi = \frac{1}{2} \tan^{-1}\left(\frac{\omega t_0}{2h^2 A}\right)$$

We will first let $A = \mu a^2/h^4$ (where μ is an arbitrary scale factor), use $x_m = 2h^2/vt_h$, and define $\rho = \omega x_m t_0/(\mu v t_h)$ to obtain (after normalization to unit dc amplitude):

$$\bar{S}(\omega,k) = \frac{e^{i\omega t_0}}{(1+\rho^2)^{1/4}} \exp\left[\frac{-x_m^2 k^2}{4(1+\rho^2)}\right] \exp\left(-\frac{i}{2}\rho \tan^{-1}\rho\right) \exp\left[\frac{i x_m^2 \rho k^2}{4(1+\rho^2)}\right]$$

The above form corresponds to small offsets (when the defined value of $A = a^2/h^4$ is large), and since ρ is proportional to h^2, it reduces to the pure shift operator $\exp(i\omega t_0)$ when $h \to 0$.

The "high" offset form (but with $h << vt_h$) is obtained letting $A \to 0$ in the original result, which yields

$$S(\omega,k) = \frac{\pi^{1/2} h\, e^{i\omega t_0}}{(2\omega t_0)^{1/2}} \cdot e^{-i\pi/4} \cdot \exp\left(\frac{i k^2 h^2}{2\omega t_0}\right)$$

where, apart from the required dip moveout term (Figure 10),

$$\Delta t_d = \frac{k^2 h^2}{2\omega^2 t_0}$$

We also have a phase shift combined with a half-integrator:

$$\frac{e^{i\pi/4}}{\sqrt{\omega}} = \frac{e^{i\pi/2}}{\sqrt{i\omega}}$$

and an amplitude decay proportional to $t_0^{1/2}$. This warns us that corresponding corrections will have to be devised for the time domain smear operator if it is to be viable at high offsets.

Figures 7-12 demonstrate the problems with the raw smear-stack operators at high offsets ($f = 40$ traces = 1000 m; $f/v = 0.4$ sec). Figure 7 shows the ideal zero-offset section (but without diffractions), and Figure 8 the constant offset section input to the operator. Figure 9 is the result of the application of the full smile, whilst in Figure 10 the operator has been artificially cut back so that the smile has half its natural horizontal extent ($x_m' = x_m/2 = f^2/4vt_h$). Note the amplitude decay with depth and the fact that the shallow reflector at small dip (10^o) suffers, if anything, greater distortion than the steeply dipping deeper reflectors. The dip moveout component may be clearly seen.

Figure 11 is a repeat for the first half-second using twice the vertical sampling rate with an impulsive reflector. Finally, Figure 12 shows corresponding mappings to zero offset of constant amplitude diffraction curves. This is clearly a less demanding test. It should be emphasized that the "dispersion" near the input level $t = f/v$ is not confined to time domain operators.

Stationary phase approach to normal and dip moveout

We start with the *double square root equation* (Claerbout, SEP-15), which was originally formulated as a downward-continuation operator [Clayton, SEP-14, Equation (5)]:

$$P(k_y,k_h,z,\omega) \quad = \quad P(k_y,k_h,0,\omega) \; \exp[i \, \frac{\omega}{v} \, \Phi \, (k_y,k_h) z]$$

We wish, however, to move away from the concept of downward-continuation with its allied imaging principles. Instead the wavefield is considered as a constant multi-dimensional block that when sliced in the (x,t)-plane yields the seismic section, whilst in the (x,z)-plane contains the reflector model. We map directly from one to the other by ray tracing via the double square root equation and the principle of stationary phase. The key idea here is that the stationary phase approximation ray traces by providing the far field solution along the ray path.

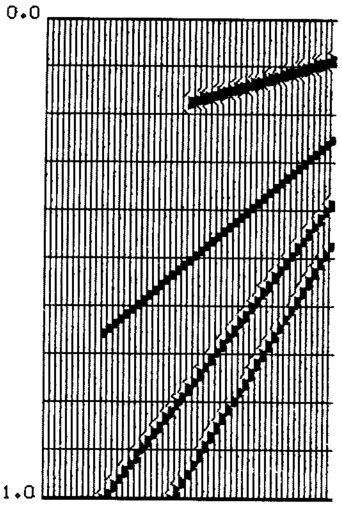


FIGURE 7.--Desired zero-offset.
δt = .01 sec.

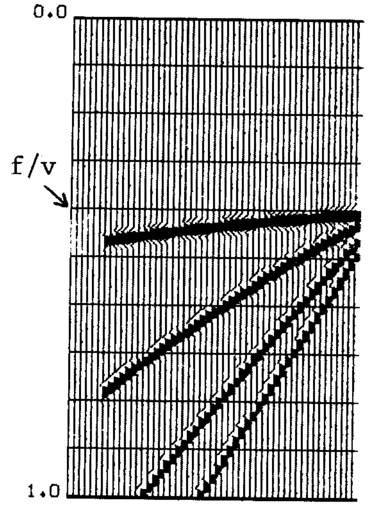


FIGURE 8.--Constant offset

dips =
10,30,50,70°

δx = 25 m

grid size =
50 x 100

offset =
1000 m =
40 traces

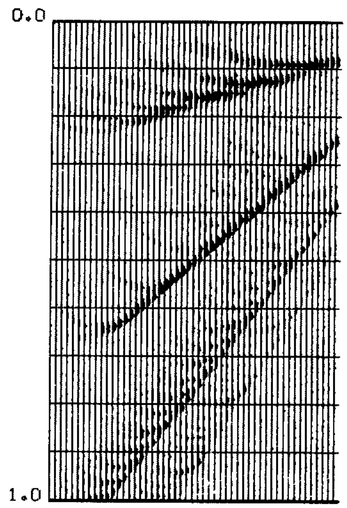


FIGURE 9.--Raw smear stack for "high" offset.

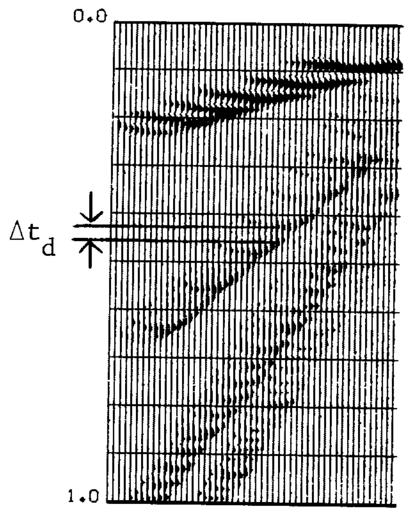


FIGURE 10.--Smear stack with reduced smile. Δt_d: dip moveout.

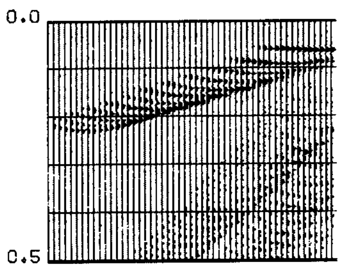


FIGURE 11a.--Half-smile.

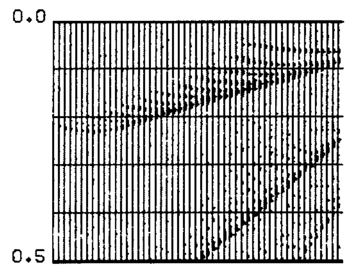
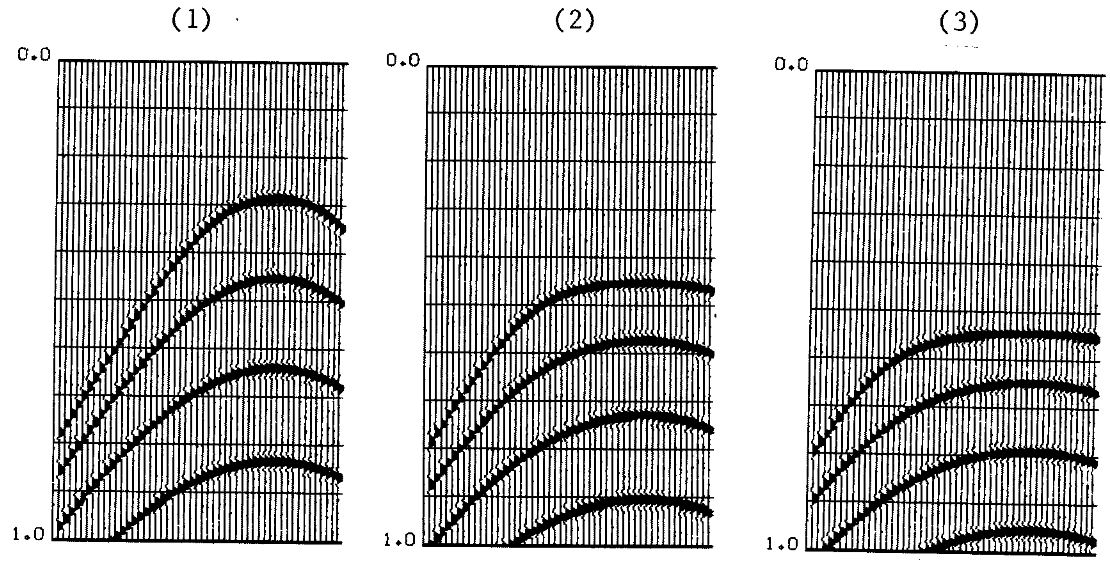


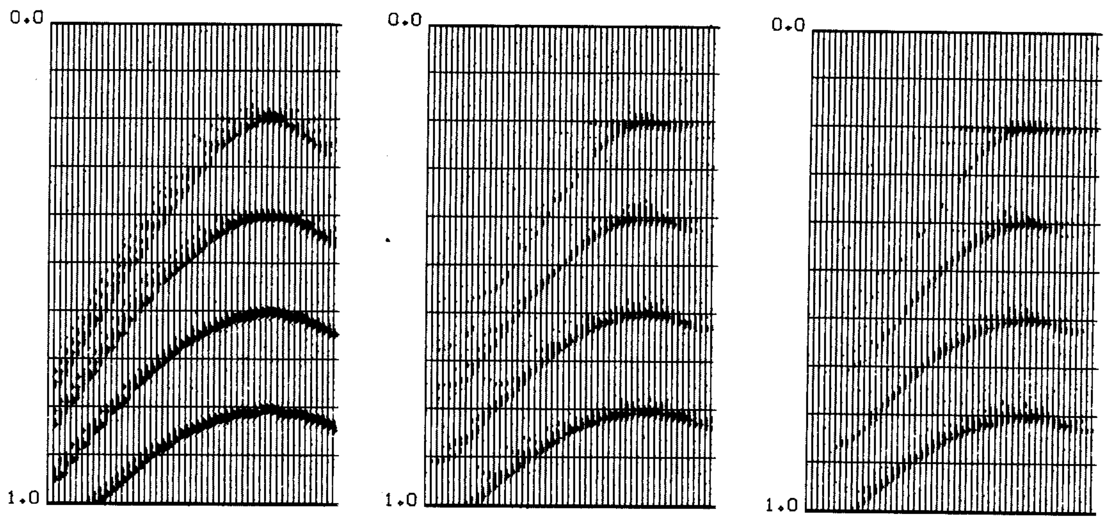
FIGURE 11b.--Full smile.

FIGURE 11.--Repeat with finer sampling.
δt = .005 .

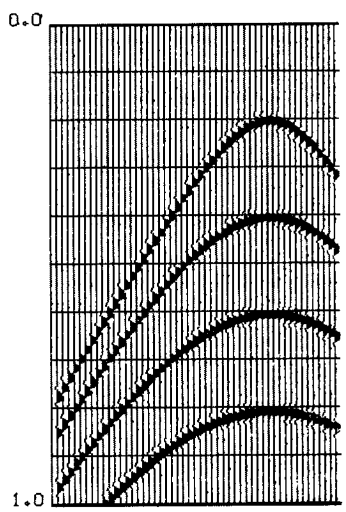
(a) Constant offset sections



(b) Smear mappings h → 0



(c) Desired zero-offset section



(1) f = 500 m = 20 traces

(2) f = 1000 m = 40 traces

(3) f = 1250 m = 50 traces

FIGURE 12.--Mapping point diffractors to zero offset.

The phase function Φ of the double square root equation is given by:

$$\Phi(k_y,k_h) = \left[1 - \frac{v^2}{\omega^2}\left(\frac{k_y + k_h}{2}\right)^2\right]^{1/2} + \left[1 - \frac{v^2}{\omega^2}\left(\frac{k_y - k_h}{2}\right)^2\right]^{1/2}$$

Direct implementation for constant offset sections therefore requires their formulation in k_y,k_h-space. As noted by Clayton (SEP-14, p. 25), the above operator is not separable in the transform domain, and hence its application would require simultaneous migration and stacking with the full three-dimensional Fourier transform. We will therefore abandon the k_h-domain and define the corresponding Green's function in terms of k_y, z and h:

$$M(k_y,z,h) = \int \exp(iz\omega\Phi/v)\ e^{-ik_h h}\ dk_h$$

where

$$\frac{z\omega}{v}\Phi = G\left(\frac{k_y + k_h}{2}\right) + G\left(\frac{k_y - k_h}{2}\right)$$

if we define

$$G(k) = \frac{z\omega}{v}\left(1 - \frac{v^2k^2}{\omega^2}\right)^{1/2}$$

We now make the substitution $k_s = (k_y - k_h)/2$ and eliminate k_h so that the Green's function is seen to correspond to a convolution product:

$$M(k_y,z,h) = -2\int \exp\{i[G(k_s) + k_s h]\} \cdot \exp\{i[G(k - k_s) - (k - k_s)h\}dk_s$$

$$= \overline{H}_1(k_s) * \overline{H}_2(k_s)$$

where $\overline{H}_{1,2}(k) = i\sqrt{2}\ \exp\{i[G(k) \pm kh]\}$

The above corresponds to two ray-tracing components, one for the shot at surface position $y - h$ and the other for the geophone at $y + h$. In the y-domain, this convolution becomes the direct product of appropriate Hankel functions: $H(y-h,z,\omega) \cdot H(y+h,z,\omega)$.

This could be computationally feasible, but let us instead return to the Green's function and perform the integration according to the principle of stationary phase (Born and Wolf, Appendix III) to derive cheap wavefield continuation procedures:

$$\int \exp\,[i\ \Psi(k_s)dk_s] \approx \left(\frac{-\pi}{2\Psi''(\hat{K}_s)}\right)^{1/2} \exp\{i[\Psi(\hat{K}_s) - \pi/4]\}$$

where $\hat{K}_s(k,h)$ is found by solving $\Psi'(\hat{K}_s) = 0$.

From $\Psi(k_s) = G(k - k_s) + G(k_s) - kh + 2hk_s$, where $k = k_y$, we obtain

$$\Psi'(\hat{K}_s) = -G'(k - \hat{K}_s) + G'(\hat{K}_s) + 2h = 0$$

But

$$G'(k) = \frac{\frac{-z\omega k}{v}}{\left(1 - \frac{v^2k^2}{\omega^2}\right)^{1/2}}$$

$$= -F(k)$$

where as we have seen that $F(k)$ is just the horizontal projection of the one-way ray path to depth z. The equation to be solved for the point of stationary phase is, therefore, simply

$$2h = F(\hat{K}_s) - F(k - \hat{K}_s)$$

$$= F\left[\frac{1}{2}(k - \hat{K}_h)\right] - F\left[\frac{1}{2}(k + \hat{K}_h)\right]$$

Even in a constant velocity medium the "sideways continuation operator" cannot be solved for in terms of a general explicit expression, but has to be considered in terms of the root of a quartic. We will therefore seek approximate solutions that are appropriate when either h is small or k is small. Ultimately we might consider numerical solutions, which in fact would be the approach for general velocity variations.

We will first consider the small offset solution, which will nevertheless be correct for all angles of incidence. Taylor's expansion is employed with $F_0 = F(k/2)$ to obtain

$$2h \approx \left[F_0 - \left(\frac{1}{2}\hat{K}_h\right)F_0' + \frac{1}{2}\left(\frac{1}{2}\hat{K}_h\right)^2 F_0''\right]$$

$$- \left[F_0 + \left(\frac{1}{2}\hat{K}_h\right)F_0' + \frac{1}{2}\left(\frac{1}{2}\hat{K}_h\right)^2 F_o''\right]$$

$$= -\hat{K}_h F_0'$$

Hence,

$$\hat{K}_h \approx -\frac{2h\omega}{vz}\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{3/2}$$

or, on using $t_0 = (2z/v)\cos\theta$ (with $\sin\theta = kv/4\omega$),

$$\hat{K}_h = -\frac{4h\omega}{v^2 t_0}\left(1 - \frac{k^2v^2}{4\omega^2}\right)$$

This equation is related to the value given to $H = v\hat{K}_h/\omega$ by Clayton in SEP-14 (p. 32) and now more correctly referred to as the operator $\hat{H}$ by Yılmaz and Claerbout in this report (p. 13). In the k_h-domain the Green's function is seen to be non-zero only in the neighborhood of $k_h = \hat{K}_h(k,h)$. This tells us that we can substitute $k_h = \hat{K}_h$ in the double square root equation without any great loss of precision (the more general expression for layered media will be given later). So that we now have

$$\Psi = G\left[\frac{1}{2}(k + \hat{K}_h)\right] + G\left[\frac{1}{2}(k - \hat{K}_h)\right] - kh + (k - \hat{K}_h)h$$

$$\approx \frac{z\omega}{v}\left[2\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{1/2} + \frac{h^2}{z^2}\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{3/2}\right]$$

where Taylor's expansion has once more been employed. The phase component of the double square root equation therefore yields the operator

$$\exp(i\Psi) = \exp(i\omega\Delta t_M)$$

where

$$\Delta t_M = \frac{2z}{v}\cos\theta + \frac{h^2}{vz}\cos^3\theta = \Delta t_m + \Delta t_N$$

is the total migration moveout for small offset sections. The component moveout term,

$$\Delta t_m = \frac{2z}{v}\cos\theta = \frac{2z}{v}\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{1/2}$$

is that required for the migration of zero-offset sections as given by the telescope equation (Claerbout, SEP-11: "Migration with Fourier Transforms"). The second moveout component Δt_N represents the total normal moveout in the presence of dip and is the same as the result obtained with geometrical optics:

$$\Delta t_N = \frac{h^2}{vz}\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{3/2}$$

To see this we substitute for the vertical depth

$$z = \frac{1}{2}vt_0\cos\theta = \frac{1}{2}vt_0\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{1/2}$$

and obtain

$$\Delta t_N = \frac{2h^2}{v^2t_0} - \frac{h^2k^2}{2\omega^2t_0}$$

$$= \Delta t_n + \Delta t_d$$

where Δt_n is the normal moveout and Δt_d the dip moveout (Figure 10).

The amplitude component is also interesting:

$$L(k,\omega) = -\left(\frac{-8\pi}{\Psi''}\right)^{1/2} e^{-i\pi/4}$$

where $\Psi'' \approx 2F_0' + (1/4)(F_0''\hat{K}_h^2)$. We drop the second order term to obtain

$$\Psi'' \approx -\frac{2vz}{\omega}\left(1 - \frac{k^2v^2}{4\omega^2}\right)^{3/2}$$

and hence,

$$L(k,\omega) \approx -2\left(\frac{\pi\omega}{vz}\right)^{1/2} \left(1 - \frac{k^2v^2}{4\omega^2}\right)^{3/4} e^{-i\pi/4}$$

The point of interest in the above is the dip component $\cos^{3/2}\theta$ which is incorrectly absent in the case of the exploding reflector model. This dip-dependence arises as the use of the double square root equation actually traces the ray path from source to reflector and then up to the receiver. Although such dip-dependence is to be expected, it should be remembered that the double square root equation corresponds to a downward-continuation operator and consequently becomes the identity operator at $z = 0$ (Claerbout, SEP-14, "Downward Continuing Constant Offset Sections a Paradox and Four Guesses"). This indicates a built-in directionality, and consequently the above dip-dependence may be stronger than required.

We now return to the ray-trace expression for $\hat{K}_h$ and derive corresponding expressions for all h under the assumption that k is small. This produces a 15-degree operator for constant offset sections:

$$\hat{K}_h \approx -\frac{2\omega}{v}\frac{1}{\left(1+\frac{z^2}{h^2}\right)^{1/2}} + \frac{3}{4}\frac{v}{\omega}\frac{\left(1+\frac{z^2}{h^2}\right)^{1/2}}{\frac{z^2}{h^2}}k^2 + 0(k^4)$$

The phase component now yields the operator

$$M(z,k,h,\omega) = \exp(i\Psi) = \exp(i\omega\Delta t_M)$$

with the migration moveout given by

$$\Delta t_M = \frac{2}{v}(z^2+h^2)^{1/2} - \frac{k^2v(h^2+z^2)^{3/2}}{4\omega^2z^2} + 0(k^4)$$

Inspection of the above expression shows that at large offsets and small z this operator has a large spatial frequency k^2 component: this corresponds to the fact that the traveltime curves on high offset sections are flat over the horizontal interval between the shot/geophone pair and then proceed to rapidly curve away (Figure 12), and also explains the poor behavior of the smear stack when $z \ll f$, as in the case of the top reflector of Figure 9. Further, it is interesting to note that (neglecting amplitudes)

$$\hat{K}_z = \partial_z \Psi$$

$$= \frac{2\omega}{v\left(1 + \frac{h^2}{z^2}\right)^{1/2}} - \frac{k^2 v}{4\omega}\left(1 + \frac{h^2}{z^2}\right)^{1/2}\left(1 - \frac{2h^2}{z^2}\right)$$

To tie in with the exploding reflector model, we introduce the pseudo-velocity $c = (1/2)v$, so that

$$\hat{K}_z = \frac{\omega}{c\left(1 + \frac{h^2}{z^2}\right)^{1/2}} - \frac{k^2 c}{2\omega}\left(1 + \frac{h^2}{z^2}\right)^{1/2}\left(1 - \frac{2h^2}{z^2}\right)$$

We have therefore managed to transform the original equation for k_z expressed in terms of the normalized wavenumber H into one (for $\hat{K}_z$) involving the half-offset h. This gives a useful starting point for Devilish-type operators (Claerbout, Yilmaz, this report). The above equations have been derived in terms of the vertical depth z and therefore correspond to modeling. To obtain the migration forms, we simply substitute $z = (1/2)vt_0 \cos\theta$.

Layered media

The previous discussion has been in the context of a constant velocity medium. We will now return to the double square root equation and extend the formulation to deal with a layered earth. In this case,

$$k_z = \left[\frac{\omega^2}{v^2(z)} - \frac{1}{2}(k - k_h)^2\right]^{1/2} + \left[\frac{\omega^2}{v^2(z)} - \frac{1}{2}(k + k_h)^2\right]^{1/2}$$

and the corresponding "telescope" transfer function

$$\exp(ik_z z)$$

becomes $\exp\left(\int_0^z k_z \, d\zeta\right)$

The Green's function now takes the form

$$-2\int_{-\infty}^{\infty} \exp\left[i\left(\int_0^z \left\{\left[\frac{\omega^2}{v^2} - (k - k_s)^2\right]^{1/2} + \left(\frac{\omega^2}{v^2} - k_s^2\right)^{1/2}\right\} d\zeta - k_s h + (k - k_s)h\right)\right] dk_s = -2\int_{-\infty}^{\infty} \exp[i\Psi(k_s)]dk_s$$

We again apply the principle of stationary phase and use the functions:

$$F(k) = \int_0^z \frac{\nu k}{(1 - \nu^2 k)^{1/2}} d\zeta \; ; \qquad \nu = \frac{v(z)}{\omega}$$

$$G(k) = \int_0^z \frac{1}{\nu} (1 - \nu^2 k^2)^{1/2} \, d\zeta$$

so that $G'(k) = -F(k)$, where $F(k)$ is still the horizontal projection of the ray path (Grant and West, 1965, p. 133). If we assume small h, then from the Taylor expansion, applied to the stationary phase result, we obtain to fourth order

$$\hat{K}_h \approx -\frac{2h}{F_0'} + \frac{h^3 F_0'''}{3(F_0')^4}$$

for the side-ways migration operator, and hence

$$\Psi = 2G_0 + \frac{h^2}{F_0'} - \frac{h^4 F_0'''}{12(F_0')^4} + 0(h^6)$$

for the corresponding phase function, which in turn defines a total migration moveout of

$$\Delta t_M = \frac{2G_0}{\omega} + \left[\frac{h^2}{\omega F_0'} - \frac{h^4}{12\omega}\frac{F_0'''}{(F_0')^4}\right]$$

$$= \Delta t_m + \Delta t_N$$

This formula is the main result of the paper and allows the migration and modeling of non-zero-offset sections. The migration moveout for zero offset is once more Δt_m:

$$\Delta t_m = \frac{2G_0}{\omega} = 2\int_0^z \frac{\left(1 - \frac{v^2(\zeta)k^2}{4\omega^2}\right)^{1/2}}{v(\zeta)} d\zeta$$

$$= 2\int_0^z \frac{\cos\theta(\zeta)}{v(\zeta)} d\zeta$$

The corresponding expression for the normal and dip moveouts is

$$\Delta t_N = \frac{h^2}{\omega F_0'} - \frac{h^4 F_0'''}{12\omega(F_0')^4}$$

where

$$F_0' = \frac{1}{\omega}\int_0^z \frac{v\, d\zeta}{\cos^3\theta} = \frac{1}{\omega}\int_0^z \frac{v(\zeta)\, d\zeta}{\left(1 - \frac{k^2v^2(\zeta)}{4\omega^2}\right)^{3/2}}$$

and

$$F_0''' = \frac{3}{\omega^3}\int_0^z \frac{v^3(5 - 4\cos^2\theta)}{\cos^7\theta} d\zeta$$

$$= \frac{3}{\omega^3}\int_0^z \frac{v^3(\zeta)\left(1 + \frac{k^2v^2(\zeta)}{\omega^2}\right)}{\left(1 - \frac{k^2v^2(\zeta)}{4\omega^2}\right)^{7/2}} d\zeta$$

The sideways continuation operator may now be written as

$$\hat{K}_h = -\frac{2h\omega}{\int_0^z \left(\frac{v}{\cos^3\theta}\right) d\zeta} + \frac{\omega h^3 \int_0^z \frac{v^3(5 - 4\cos^2\theta)}{\cos^7\theta} d\zeta}{\left[\int_0^z \left(\frac{v}{\cos^3\theta}\right) d\zeta\right]^4}$$

Notice that, for larger $v(z)$, F_0' is limited for an ever-decreasing range of k's so that $\hat{K}_h$ is eventually non-zero for only the lowest wavenumbers. For small k we in fact obtain

$$F_0' = \frac{1}{\omega}\int_0^z \frac{v\,d\zeta}{\cos^3\theta} = \frac{1}{\omega}\int_0^{t_0} \frac{v^2 dt}{2\cos^2\theta}$$

$$\approx \frac{v_{rms}^{\ 2}\ t_0}{2\omega}\left(1 + \frac{k^2\overline{v^4}}{4\omega^2 v_{rms}^{\ 2}}\right)$$

The total normal moveout therefore becomes

$$\Delta t_N \approx \frac{h^2}{\omega F_0'} \approx \frac{2h^2}{v_{rms}^{\ 2} t_0}\left(1 - \frac{k^2\overline{v^4}}{4\omega^2 v_{rms}^{\ 2}}\right)$$

$$= \frac{2h^2}{v_{rms}^{\ 2} t_0} - \frac{h^2 k^2}{2\omega^2 t_0} M^2$$

where

$$\overline{v^4} = \frac{1}{t_0}\int_0^{t_0} v^4\,dt; \qquad v_{rms}^{\ 2} = \frac{1}{t_0}\int_0^{t_0} v^2\,dt$$

and

$$M^2 = \frac{\overline{v^4}}{v_{rms}^{\ 4}}$$

We therefore see that the smear-stack approach also holds in layered media albeit with a different interpretation of the terms.

So far we have sought approximate analytical solutions to the ray trace equation

$$2h = F\left[\frac{1}{2}(k - \hat{K}_h)\right] - F\left[\frac{1}{2}(k + \hat{K}_h)\right]$$

$$= \frac{1}{2}\int_0^z \left\{ \frac{(k - \hat{K}_h)\nu(\zeta)}{\left[1 - \frac{\nu^2(\zeta)(k - \hat{K}_h)^2}{4}\right]^{1/2}} - \frac{(k + \hat{K}_h)\nu(\zeta)}{\left[1 - \frac{\nu^2(\zeta)(k + \hat{K}_h)^2}{4}\right]^{1/2}} \right\} d\zeta$$

where $\nu(z) = v(z)/\omega$. Besides actually determining the Green's function as a convolution, we could also attempt a numerical solution of the above equation for $\hat{K}_h$. For this purpose it is rewritten as

$$\int_0^z Q[\zeta,k,\hat{K}_h(z)]\,d\zeta = 2h$$

and differentiation with respect to z gives us

$$\frac{d\hat{K}_h}{dz}\int_0^z \frac{\partial Q}{\partial \hat{K}_h}\,d\zeta + Q = 0$$

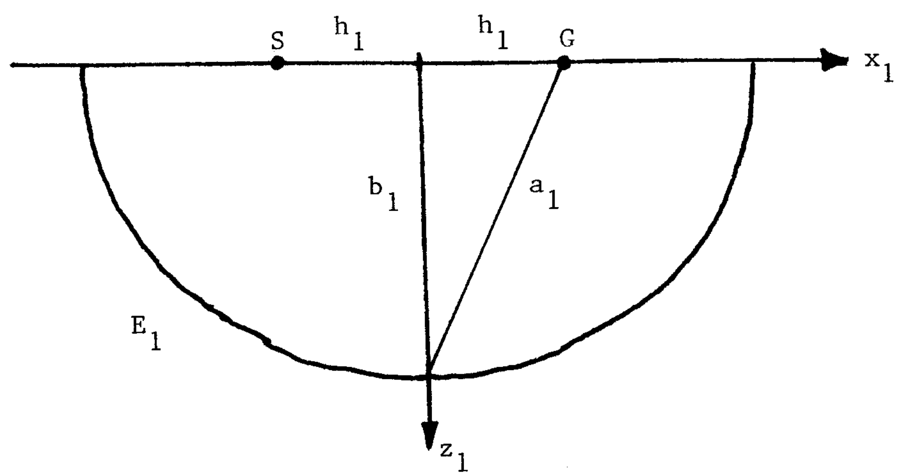
or

$$\frac{d\hat{K}_h}{dz} = \frac{-Q}{\int_0^z Q'\,d\zeta}$$

We now solve the above equation at each level by, say, an iterative ray tracing method for $\hat{K}_h$. Such numerical techniques would have to be used in the presence of lateral velocity variations.

Further geometric optics: Changing offsets

In the first section an approximate smear operator was derived for mapping constant offset sections to zero offset. We will now follow a similar argument to derive the operator required for transferring from one offset h_1 to another h_2.

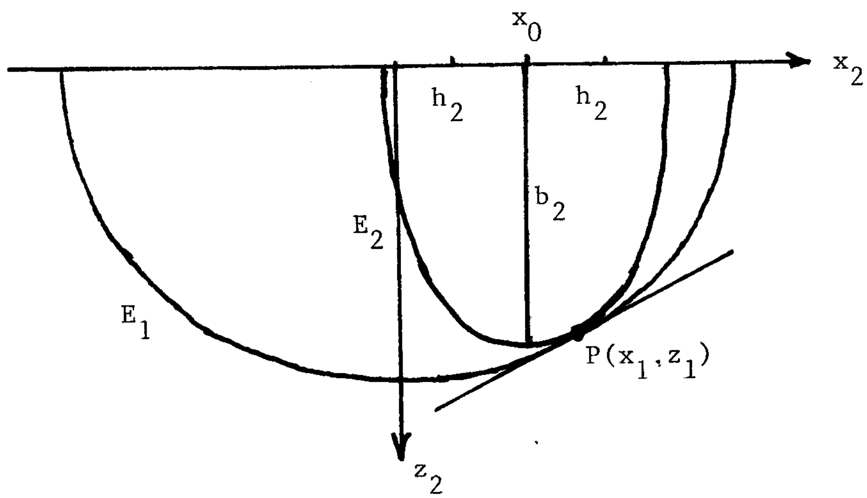


Let the elliptical reflector corresponding to an impulse on the constant offset time section with half-offset h_1 be given by

$$z_1 = b_1 \sin\phi_1; \quad x_1 = (b_1^2 + h_1^2)^{1/2}\cos\phi_1$$

Each point (x_1, z_1) on ellipse E_1 is now replaced by a tangential ellipse E_2 which corresponds to a half-offset h_2 and whose center is displaced to x_0:

$$z_2 = b_2 \sin \phi_2 ; \qquad x_2 = (b_2^2 + h_2^2)^{1/2} \cos \phi_2 + x_0$$



As P is common to both ellipses:

$$b_2 \sin \phi_2 = b_1 \sin \phi_1 \tag{1}$$

and

$$(b_2^2 + h_2^2)^{1/2} \cos \phi_2 + x_0 = (b_1^2 + h_1^2)^{1/2} \cos \phi_1 \tag{2}$$

whilst tangency gives

$$\frac{(b_1^2 + h_1^2)^{1/2}}{b_1} \tan \phi_1 = \frac{(b_2^2 + h_2^2)^{1/2}}{b_2} \tan \phi_2 \tag{3}$$

We first eliminate ϕ_2 between (1) and (2) and obtain an equation for b_2 of which the required root is

$$b_2^2 = \frac{B + [B^2 + 4(b_1^2 + h_1^2)h_2^2 b_1^2 \cos^2 \phi_1]^{1/2}}{2(b_1^2 + h_1^2)}$$

where

$$B = b_1^4 + b_1^2 h_1^2 \sin^2 \phi_1$$

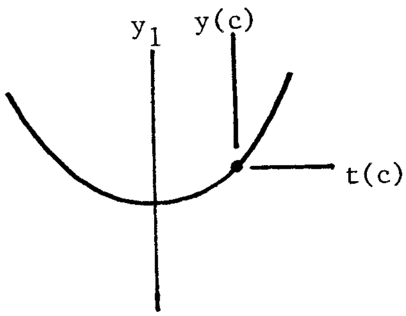
and from (2):

$$x_0 = (b_1^2 + h_1^2)^{1/2} \cos \phi_1 - (b_2^2 + h_2^2)^{1/2} \cos \phi_2$$

But from (1):

$$\cos \phi_2 = \left(1 - \frac{b_2^2 \sin^2\phi_1}{b_1^2}\right)^{1/2}$$

Hence, $x_0 = (b_1^2 + h_1^2)^{1/2} \cos \phi_1 - \frac{1}{b_1} [(b_2^2 + h_2^2)(b_1^2 - b_2^2 \sin^2\phi_1)]^{1/2}$



The above equations can now be used to define a smear operator by virtue of which each point at (y_1, t_1) on the section with offset h_1 is smeared over the parametric curve $[y(c), t(c), |c| < 1]$ centered at y_1 $(c \equiv \cos \phi_1)$ and defined according to

$$y(c) = y_1 + x_0(c)$$

$$t(c) = \frac{2}{v}\left(b_2^2(c) + h_2^2\right)^{1/2}$$

with b_2 defined as above, and $b_1^2 = v^2 t_1^2/4 - h_1^2$.

It is now possible to formulate an operator which maps the section with offset f_{max} to the next lower offset $f_{max} - \Delta f$ and therefore allows these two sections to be stacked. This step is now repeated up to zero offset, thus producing a "sideways migration." The velocity model may be defined en route according to that which gives the best stack at each stage.

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GRANT and WEST (1965), *Interpretation Theory in Applied Geophysics* (New York: McGraw-Hill).

APPENDIX I - *Definitions and Notation*

t	Two-way traveltime as read down trace on a raw gather $t = t_s + t_g$. This is the time as given by the real space line integral along a curve defined to follow the ray path from a shot S to a reflector/diffractor point P and then up to the geophone G.
t_h	t for a constant offset gather, with half-offset h.
t_0	Two-way traveltime for zero offset $t_0^2 \approx t_h^2 - f^2/v^2$.
t_s	Traveltime from shot to reflector/diffractor.
t_g	Traveltime from geophone to reflector/diffractor.
t_z	Vertical two-way traveltime: as read off a migrated section.
τ	One-way vertical traveltime = $(1/2)t_z$.
v	True velocity of medium.
c	Half-velocity of medium = $(1/2)v$. Used in formulating "exploding reflector" equations.
x	Horizontal coordinate as measured along a seismic line. Also the horizontal coordinate of a zero-offset section.
x_s, x_g	The horizontal coordinates of the shot, geophone respectively.
y	Midpoint coordinate $y = (1/2)(x_s + x_g)$
f	Full offset $x_g - x_s$
h	Half-offset = $(1/2)f$.
Δt_M	Total migration moveout to be used in mapping from a common offset gather to a migrated section. This mapping may be symbolically written $(y,t) \rightarrow (x,t_z)$. It can be thought of as the offset, time, position and velocity-dependent shift required when performing the appropriate smear-stacks. In the transform domain, it takes the form $M = \exp(i\omega\Delta t_M)$.
Δt_m	The moveout appropriate to the migration of zero-offset section $\approx 2z(1 - k^2v^2/4\omega^2)^{1/2}/v$.
Δt_n	Normal moveout (ignoring dip) $\approx f^2/2v^2t_0$
Δt_d	Dip moveout $\approx 2h^2 \sin^2\alpha/v^2t_0$; $h^2k^2/2t_0\omega^2$
Δt_N	Total normal moveout (with dip) $\approx \Delta t_n + \Delta t_d$
k_s, k_g	Horizontal wavenumbers associated with shot and geophone positions, respectively.

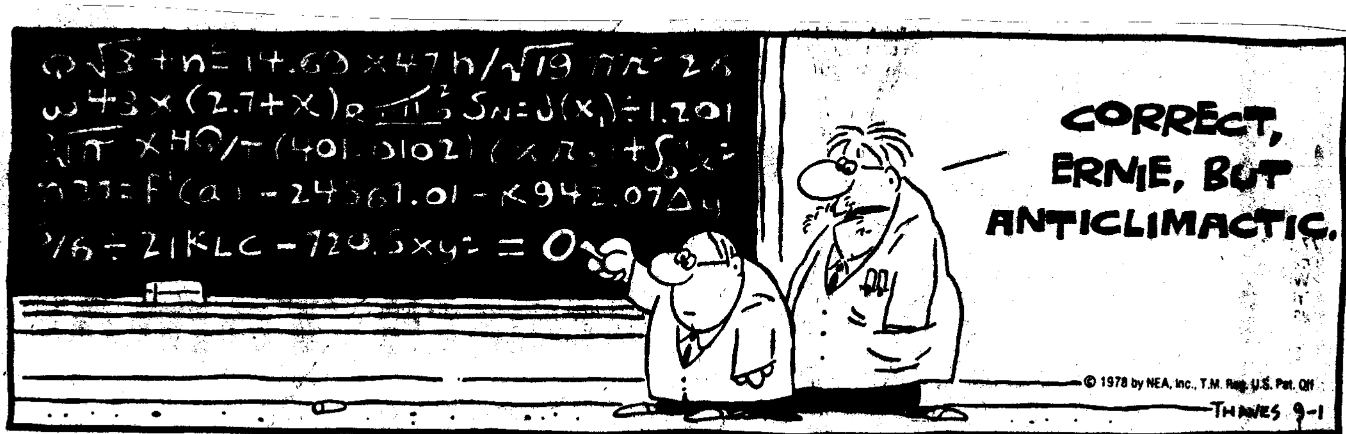
k_y	Wavenumber associated with midpoint coordinate $k_y = k_s + k_g$.
k_h	Wavenumber associated with half-offset coordinate $k_h = k_g - k_s$.
k_z	Vertical wavenumber.
k	Used for k_y in the k_y,h domain.
$\hat{K}_s,\hat{K}_h,\hat{K}_z$	Functions which map the corresponding wavenumbers into the h,k domain. Defined via the stationary phase equations.
α	Angle of dip of planar reflector element. The normally reflected ray adopts the wavenumber $k = \omega \sin \alpha / v$.
γ_s,γ_g	Angles of emergence from shot and geophone positions respectively. $\sin \gamma_s = k_s v/\omega$; $\sin \gamma_g = k_y v/\omega$.
θ	Angle of emergence from a coincident shot/geophone pair in zero-offset two-way traveltime geometry: $\sin \theta = kv/2\omega$.
F.T	Used for "the Fourier Transform of."
$F(k)$	The horizontal projection of the distance traversed by a ray with associated wavenumber k (Snell parameter $p = k/\omega$).

$$F(k) = \int_0^z \frac{v(\zeta)p\, d\zeta}{(1 - v^2p^2)^{1/2}} = \int_0^z \frac{\frac{v^2k^2}{\omega^2}}{\left(1 - \frac{v^2k^2}{\omega^2}\right)^{1/2}} d\zeta$$

F_0	Used for $F[(1/2)k]$: the zero-offset result.
$G(k)$	Allied to Δt_m.

$$G(k) = \int_0^z \frac{\omega}{v}\left(1 - \frac{v^2k^2}{\omega^2}\right)^{1/2} dz \quad ; \quad F(k) = -G'(k)$$

G_0	Used for $G[(1/2)k]$.
x_m	Extremum of the smear operator which maps to zero offset; $x_m = f^2/2vt_h$.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