AN EXACT FACTORIZATION OF THE ELASTIC WAVE EQUATION

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In previous SEP reports several approximate forms of the one-way elastic displacement wave equation were derived. One gap in the theory, however, was the lack of an exact one-way operator. In this paper such an operator is derived.

The analogy of this result for the scalar wave equation is the square root operator for one-way waves

$$\left\{ \partial_z - i \frac{\omega}{v} \left[1 - \left(\frac{vk_x}{\omega} \right)^2 \right]^{1/2} \right\} P = 0 \qquad (1)$$

In scalar theory, rational approximations of the dispersion relation for exact one-way waves relate directly to the differential operators via a Fourier transform. In elastic theory, the situation is not as simple because the dispersion relations do not specify the differential operators. With an exact elastic operator available, however, the approximations can be made directly to it.

In the last section of the paper, some approximations to the exact operator are discussed.

Derivation of the exact operator

To start, the full elastic displacement equation is written in the form

$$\begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} \underline{u}_{zz} + \begin{bmatrix} 0 & \alpha^2-\beta^2 \\ \alpha^2-\beta^2 & 0 \end{bmatrix} \underline{u}_{xz} + \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} \underline{u}_{xx} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}_{tt} = 0 \qquad (2)$$

where

$$\underline{u} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \text{horizontal disp.} \\ \text{vertical disp.} \end{bmatrix}$$

and α and β are the compressional and shear velocities respectively. After Fourier transforming in z, x, and t, and defining $Z = k_z/\omega$ and $X = k_x/\omega$, the equation becomes

$$\begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} Z^2 + \begin{bmatrix} 0 & \alpha^2-\beta^2 \\ \alpha^2-\beta^2 & 0 \end{bmatrix} XZ + \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} X^2 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \tag{3}$$

This equation is now put in the form of a quadratic in Z [defined as $B(Z)$] by multiplying by the inverse of the leading coefficient.

$$B(Z) = IZ^2 + \begin{pmatrix} 0 & \dfrac{\alpha^2-\beta^2}{\beta^2} X \\ \dfrac{\alpha^2-\beta^2}{\alpha^2} X & 0 \end{pmatrix} Z + \begin{pmatrix} \dfrac{\alpha^2 X^2-1}{\beta^2} & 0 \\ 0 & \dfrac{\beta^2 X^2-1}{\alpha^2} \end{pmatrix} = 0 \tag{4}$$

The wave equation in terms of $B(Z)$ is

$$B(Z)\,\hat{\underline{u}} = 0 \tag{5}$$

where $\hat{\underline{u}}$ is the Fourier transform of $\underline{u}$. The dispersion relation for the full wave equation is given by

$$\det B(Z) = 0 \tag{6}$$

Separating the wave equation into up- and downgoing components amounts to factoring the polynomial of Equation (4) into the form

$$B(Z) = (IZ - C_l)\,(IZ - A_u) \tag{7}$$

and

$$B(Z) = (IZ - C_2)\,(IZ - A_d) \tag{8}$$

where A_u and A_d are the operators for up- and downgoing waves respectively. The operators C_1 and C_2 have no special meaning and will be ignored. Note that $C_1 \neq A_d$ and $C_2 \neq A_u$, because the factors in Equations (7) and (8) do not commute. The wave equation for upcoming waves is then

$$(IZ - A_u)\,\underline{\hat{u}} = 0 \tag{9}$$

and its dispersion relation is

$$\det\,(IZ - A_u) = 0 \tag{10}$$

Similar relations hold for the downgoing waves. In Figure 1, the dispersion relations for up- and downgoing waves are shown. They are, as one would expect, semicircles for both P and S waves.

To factor $B(Z)$ into the form of Equation (8), a technique given by Claerbout (1966) for factoring multichannel time series is used. We will restrict our attention to the downgoing wave equation, but the results for upgoing waves follow in an analogous manner. From Cramer's rule for matrix inverses, we have

$$B^{-1} = \frac{\text{adj } B}{\det B}$$

or

$$B^{-1} \det B = \text{adj } B \tag{11}$$

Post-multiplying Equation (8) by Equation (11) leads to

$$\det B = (IZ - C_2)\,(IZ - A_d)\,\text{adj } B \tag{12}$$

The determinant of B will be zero whenever we choose Z to lie on the dispersion relation of the full wave equation. For the modeling of downgoing waves the appropriate choices of Z are

$$Z = \left(\frac{1}{\beta^2} - X^2\right)^{1/2} = S_\beta \tag{13}$$

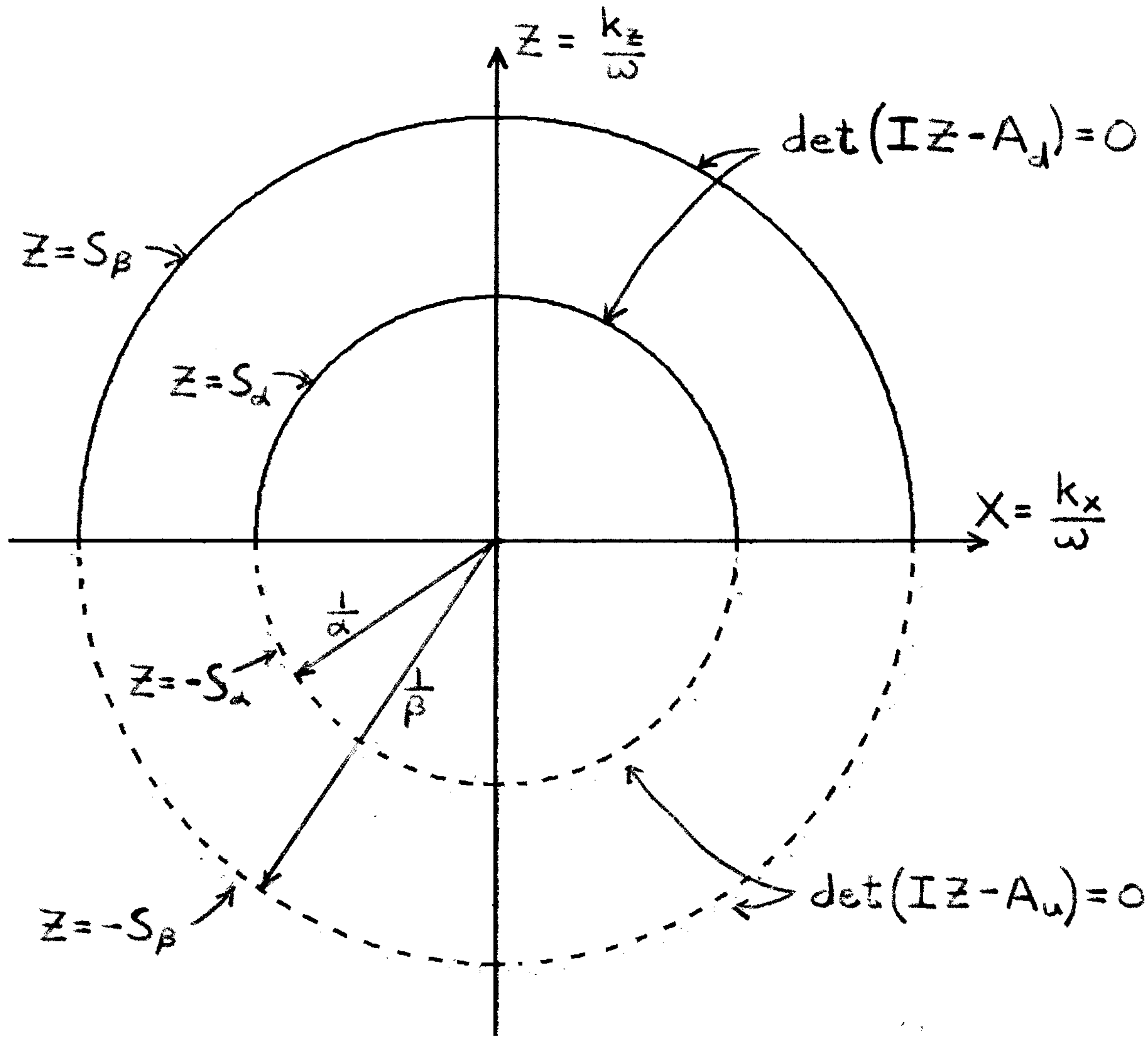

FIGURE 1.--*The dispersion relations of up- and downgoing elastic waves.* The solid semi-circles is the dispersion relation of the downgoing elastic wave operator. The outer semi-circle ($Z = S_\beta$) has a radius of $1/\beta$, and represents downgoing shear waves. The inner solid semi-circle ($Z = S_\alpha$) is for compressional waves, and has a radius of $1/\alpha$. The dashed semi-circles is the dispersion of upgoing waves and also has shear and compressional components.

and

$$Z = \left(\frac{1}{\alpha^2} - X^2 \right)^{1/2} = S_\alpha \tag{14}$$

Whenever the determinant of a 2 x 2 matrix is zero, the adjoint of that matrix can be factored into the product of a column vector (c), and a row vector (r) (*i.e.* the matrix is rank 1). Thus for $Z = S_\beta$, Equation (12) reduces to

$$\det B(S_\beta) = (IS_\beta - C_2)\ \underline{(IS_\beta - A_d)\ c}\ r^T = 0 \tag{15}$$

The underscored part of this equation has the form of an eigenproblem, with S_β as an eigenvalue and c as an eigenvector of the matrix A_d. If we had evaluated the determinant at $Z = S_\alpha$, then we would obtain another eigenvector corresponding to the eigenvalue S_α. Knowledge of the eigenvectors and eigenvalues allows us to construct A_d from the decomposition

$$A_d = Q^{-1} \Lambda Q \tag{16}$$

where Λ is a diagonal matrix of the eigenvalues and Q is a matrix whose columns are the eigenvectors.

Proceeding with the above arguments to find A_d, we have for $Z = S_\beta$

$$\operatorname{adj} B(S_\alpha) = (\alpha^2 - \beta^2) \begin{bmatrix} \frac{1}{\alpha^2} S_\beta{}^2 & \frac{-X}{\beta^2} S_\beta \\ \frac{-X}{\alpha^2} S_\beta & \frac{1}{\beta^2} X^2 \end{bmatrix}$$

$$= (\alpha^2 - \beta^2) \begin{bmatrix} -S_\beta \\ X \end{bmatrix} \begin{bmatrix} \frac{-S_\beta}{\alpha^2}, & \frac{X}{\beta^2} \end{bmatrix} \tag{17}$$

Hence, the eigenvector corresponding to the eigenvalue S_β is $(-S_\beta, X)^T$. For $Z = S_\alpha$ we have

$$\text{adj } B(S_\alpha) = (\alpha^2 - \beta^2) \begin{pmatrix} \dfrac{-X^2}{\alpha^2} & \dfrac{-XS_\alpha}{\beta^2} \\ & \\ \dfrac{-XS_\alpha}{\alpha^2} & \dfrac{-S_\alpha{}^2}{\beta^2} \end{pmatrix}$$

$$= (\alpha^2 - \beta^2) \begin{bmatrix} X \\ S_\alpha \end{bmatrix} \begin{bmatrix} \dfrac{-S_\alpha}{\alpha^2} , \dfrac{X}{\beta^2} \end{bmatrix} \tag{18}$$

The eigenvector for S_α is $(X, S_\alpha)^T$. Finally, constructing A_d from Equation (16) with

$$\Lambda = \begin{bmatrix} S_\beta & 0 \\ 0 & S_\alpha \end{bmatrix}, \qquad Q = \begin{bmatrix} -S_\beta & X \\ X & S_\alpha \end{bmatrix} \tag{19}$$

we have

$$A_d = \frac{1}{X^2 + S_\alpha S_\beta} \begin{bmatrix} \dfrac{1}{\beta^2} S_\alpha & XS_\beta(S_\alpha - S_\beta) \\ XS_\alpha(S_\alpha - S_\beta) & \dfrac{1}{\alpha^2} S_\beta \end{bmatrix} \tag{20}$$

To obtain the upgoing wave equation, the analysis of Equations (16) through (21) is repeated with eigenvalues of $-S_\beta$ and $-S_\alpha$. The result is

$$A_u = \frac{1}{X^2 + S_\alpha S_\beta} \begin{bmatrix} \dfrac{-1}{\beta^2} S_\alpha & XS_\beta(S_\alpha - S_\beta) \\ XS_\alpha(S_\alpha - S_\beta) & \dfrac{-1}{\alpha^2} S_\beta \end{bmatrix} \tag{21}$$

A comparison of the one-way elastic operators with the scalar operator (Equation 1), shows the situation is much more complicated for elastic waves. Part of the complexity is due to the fact that a displacement form of the elastic wave equation is used. Intuitively one would expect the elastic operators to decouple into two scalar problems, one for compressional waves, and one for shear waves. This expectation is true, and the decoupling is given by Equation (16). The downgoing wave equation is

$$(IZ - Q^{-1} \Lambda Q) \, \underline{\hat{u}} = 0$$

or

$$Q^{-1} (IZ - \Lambda) Q \, \underline{\hat{u}} = 0 \tag{22}$$

With the definition that $\underline{\phi} = Q \, \underline{\hat{u}}$, Equation (22) reduces to two decoupled scalar problems. The components of $\underline{\phi}$ are

$$\phi_1 = -S_\beta \, u + X \, w \tag{23}$$

$$\phi_2 = X \, u + S_\alpha \, w \tag{24}$$

Both S_α and S_β act as vertical wavenumbers and consequently are related to z-derivatives. The horizontal wavenumber X is related to an x-derivative. Consequently, the ϕ_1 component has the form of curl u or shear waves, and the ϕ_2 component has the form of div u or compressional waves.

Approximations of the exact one-way equations

In order to convert the exact one-way elastic operators into differential equations, it is necessary to approximate them with a rational power series in X. The most direct method of doing this is to expand each element in the matrix in a Taylor Series about $X = 0$. Unlike the scalar case, the expansion will involve both odd and even powers of X. The fourth order expansion of S_v where v is either α or β, is

$$S_v \sim \frac{1}{v} - \frac{v}{2} X^2 - \frac{v^3}{8} X^4 \tag{25}$$

This expansion is substituted for every square in the operator, and all terms of X^5 or higher are dropped. Note that the expression appearing in the denominator of the elements is expanded as

$$\frac{1}{X^2 + S_\alpha S_\beta} \sim \alpha\beta \left[1 + \frac{1}{2} (\beta-\alpha)^2 X^2 - \frac{(\beta-\alpha)^2}{8} (\alpha^2+\beta^2-6\alpha\beta) X^4 \right] \tag{26}$$

The resulting approximation for the downgoing operator is

$$IZ - \sum_{i=0}^{n} B_i X^i + 0(X^{n+1}) \tag{27}$$

where

$$B_0 = \begin{bmatrix} \frac{1}{\beta} & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} \qquad B_1 = (\beta - \alpha)\begin{bmatrix} 0 & \frac{1}{\beta} \\ \frac{1}{\alpha} & 0 \end{bmatrix}$$

$$B_2 = \frac{1}{2}\begin{bmatrix} \beta-2\alpha & 0 \\ 0 & \alpha-2\beta \end{bmatrix} \qquad B_3 = \frac{(\beta-\alpha)^2}{2}\begin{bmatrix} 0 & -\frac{\alpha}{\beta} \\ \frac{\beta}{\alpha} & 0 \end{bmatrix}$$

$$B_4 = -\frac{1}{8}\begin{bmatrix} \frac{1}{\beta}[\alpha^4 + (\alpha-\beta)^2(3\alpha^2+\beta^2-6\alpha\beta)] & 0 \\ 0 & \frac{1}{\alpha}[\beta^4 + (\alpha-\beta)^2(3\beta^2+\alpha^2-6\beta\alpha)] \end{bmatrix}$$

The coefficient matrices B_0, B_1, and B_2 agree with the previous one-way operators. In Figure 2 a plot of the dispersions of the downgoing wave operator is shown for various orders (values of n). Note that the approximations deteriorate as the velocity ratio α/β becomes large. The results for $n = 3$ are interesting in that they are worse than the $n = 2$ results. The best matching dispersion relation is for $n = 4$. The well-posedness of this operator, however, has not been determined. In scalar theory, a fourth power Taylor series expansion of the square operator leads to ill-posed differential equations (Engqüist, SEP-8). This provides some motivation for the approximations described next.

In the expansion of Equation (27), one class of terms that are not present are the terms involving the cross powers of X and Z (*i.e.* XZ and X Z). In the time-space domain these terms are derivatives of the type $\underline{u}_{xz}$ and $\underline{u}_{xxz}$. The addition of these terms will allow the degree of X in the approximations to be reduced while retaining the same order of accuracy. The general form of the new expansion is

$$\left(I + \sum_{i=1}^{P} D_i X^i\right) Z - \sum_{i=0}^{m} C_i X^i + 0(X^{m+p+1}) \tag{28}$$

To determine the coefficient matrices $[D_i]$ and $[C_i]$, Equation (28) is multiplied by the inverse of the Z-coefficient, and matched term by terms with Equation (27):

$$IZ - \left(I + \sum_{i=1}^{p} D_i X^i\right)^{-1} \left(\sum_{i=0}^{m} C_i X^i\right)$$

$$= IZ - \sum_{i=0}^{m+p} E_i X^i \tag{29}$$

The recurrence relation for the $[E_i]$ is

$$E_i = C_i - \sum_{k=1}^{p} D_k E_{i-k} \tag{30}$$

Equating the $[E_i]$ with $[B_i]$ of Equation (27) leads to

$$B_i = C_i - \sum_{k=1}^{p} D_k B_{i-k}, \; i = 0 \ldots\ldots\ldots n \tag{31}$$

For $i > m$, $C_i = 0$, and hence the $[D_i]$ are determined by the $p \times p$ matrix system

$$B_i = - \sum_{k=1}^{p} D_k B_{i-k}, \; i = m+1 \ldots\ldots\ldots\ldots n \tag{32}$$

Once the $[D_i]$ are determined, the $[C_i]$ are found in a recursive manner from Equation (31)

Taking for example the case of $p = 1$ and $m = 2$ (and, consequently, $n = 3$), the recursive equations are

$$\begin{aligned} B_0 &= C_0 \\ B_1 &= C_1 - DB_0 \\ B_2 &= C_2 - DB_1 \\ B_3 &= -DB_2 \end{aligned} \tag{33}$$

Solving these equations we have

$$
\begin{aligned}
D &= -B_3B_2^{-1} \\
C_0 &= B_0 \\
C_1 &= B_1 + DB_0 \\
C_2 &= B_2 + DB_1
\end{aligned}
\qquad (34)
$$

The coefficient matrices are

$$
D_1 = \frac{(\beta-\alpha)^2}{(\beta-2\alpha)(\alpha-2\beta)} \begin{bmatrix} 0 & \frac{\alpha}{\beta}(\beta-2\alpha) \\ \frac{-\beta}{\alpha}(\alpha-2\beta) & 0 \end{bmatrix} \qquad (35)
$$

$$
C_1 = -(\beta-\alpha) \begin{bmatrix} 0 & \frac{1}{\alpha-2\beta} \\ \beta-2\alpha & 0 \end{bmatrix}
$$

$$
C_2 = \frac{1}{2} \begin{bmatrix} (\beta-2\alpha) + \frac{(\beta-\alpha)^3}{\beta(\alpha-2\beta)} & 0 \\ 0 & (\alpha-2\beta) - \frac{(\beta-\alpha)^3}{\alpha(\beta-2\alpha)} \end{bmatrix}
$$

The results of this approximation are compared in Figure 3 with the previous approximation for $n = 2$. Note that the coefficient matrices are singular for $\alpha = 2\beta$. Hence, only two velocity ratios are presented. The new approximation appears to be worse and the reason for this is not understood. One possible explanation is that the third order Taylor series expansion, to which the new approximation was fitted, was the poorest of all the expansions presented.

The case of including a D_2X^2Z term but dropping the D_1XZ term in Equation (29) was also tested. The results shown in Figure 4 indicate that this approximation is also poor. However, at least the coefficient matrices were not singular.

It is hoped that the inclusion of both D_1XZ and D_2X^2Z terms in the expansion of Equation (29) will provide significantly better results. This should make the order of the approximation X^5, but still only involve up to second differentials in x. The algebra to find the coefficient matrices of this approximation is straightforward but immense, and has not yet been done. This equation should be the "45-degree" one-way elastic wave equation.

The ultimate hope for the approximations of the one-way elastic wave equation is to find a recursive scheme which generates all higher order approximations. This would be the equivalent of the Muir-Engquist recursion for higher order approximations of the scalar wave equation.

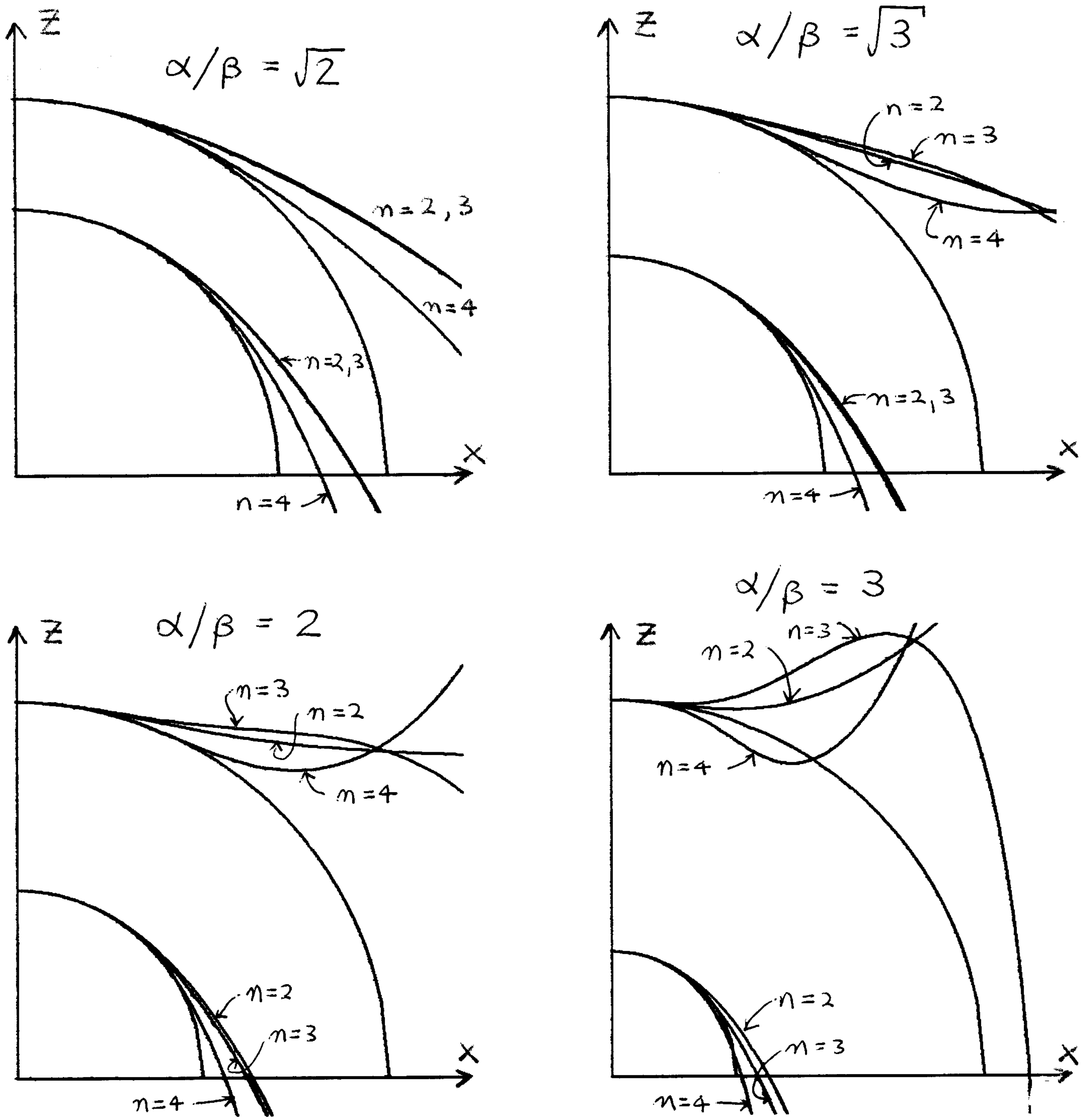

FIGURE 2.--*Taylor series expansions of the downgoing elastic wave equation.* In this figure are shown the Taylor series expansions up to fourth order for four velocity ratios. In each panel only the upper-right quarter plane of the dispersion relations are shown since the approximations are symmetric in X. The labels n = 2, n = 3, n = 4, refer to the order of the expansion displayed. The quarter circles of the exact operator are shown for reference in each panel.

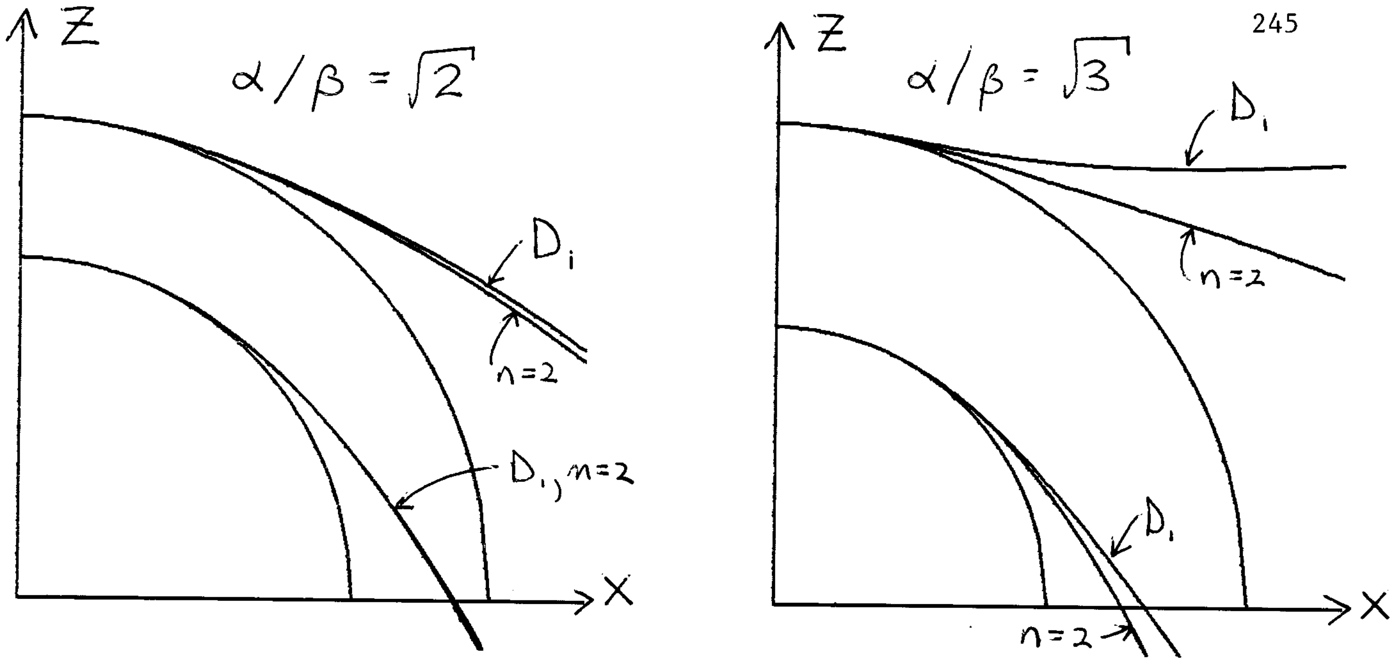

FIGURE 3.--*The dispersion relation of the approximation.*

$$(I + D_1X)Z - (C_0 + C_1X + C_2X^2)$$

The dispersion relation of this operator (labelled D) is shown for two velocity ratios (the first two cases in Figure 2). For reference, the n = 2 case from Figure 2 is repeated. The coefficient matrices in this approximation are singular at n = 2. The D approximation shown here appears poorer than the n = 2 case, despite the fact that it is of higher order.

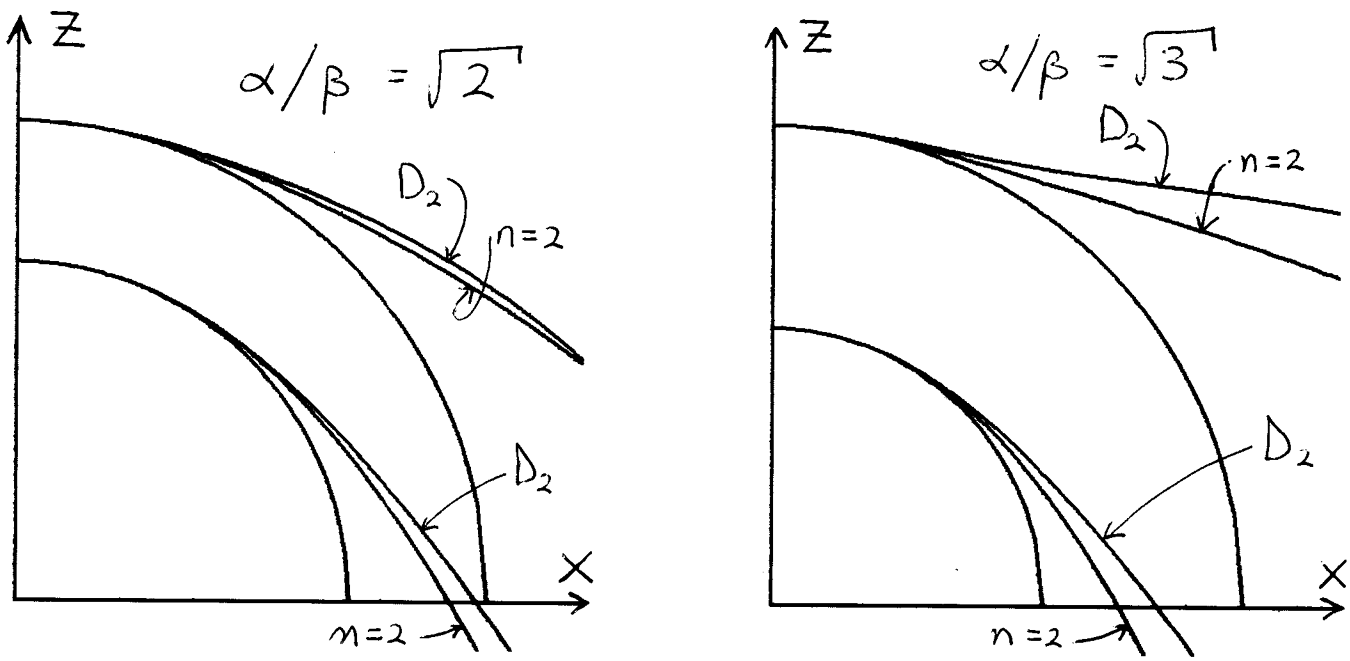

FIGURE 4.--*The dispersion relation of the approximation.*

$$(I + D_2X^2)Z - (C_0 + C_1X + C_2X^2)$$

This approximation is similar to the case shown in Figure 3, except that the D_1XZ term is replaced with a D_2X^2Z term. The coefficient matrices are no longer singular at any velocity ratios but the results are as poor as in the case in Figure 3.

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