CAUSAL POSITIVE REAL OPERATORS

David Brown

1. *The motivation for causal positive real operators*

Consider an initial-value problem for some partial differential equation, for example,

$$[\partial_z + A(\partial_x, z)]\, u = 0 \tag{1}$$

with initial values $u(x,z=0) = u_0(x)$ for $-\infty < x < \infty$ and $z \geq 0$. The function $A(\partial/\partial x, x, z)$ is a differential operator that depends on $\partial/\partial x$ but not on $\partial/\partial z$. We can go about solving this problem by first Fourier transforming over x and then integrating. Equation (1) becomes[1]

$$[\partial_z + \hat{A}(k_x, z)]\, \hat{u} = 0 \tag{2}$$

The solution to Equation (2) with the initial conditions given above is

$$\hat{u}(z,k_x) = \exp\left[-\int_0^z \hat{A}(k_x,z)\, dz\right] \hat{u}_0(k_x) \tag{3}$$

and so the solution to Equation (1) is

$$u(z,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x\, e^{ik_x x} \exp\left[-\int_0^z \hat{A}(k_x,z)\, dz\right] \hat{u}_0(k_x) \tag{4}$$

How do we know that (1) is a stable equation for integration in the positive z direction? We can answer this by looking at Equation (3), which is the solution to the Fourier transformed problem. If Equation (2) is to be a

[1]Fourier transform conventions:

$$\hat{u}(k_x) = \int_{-\infty}^{\infty} dx\, e^{-ik_x x}\, u(x); \qquad u(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} dk_x\, e^{ik_x x}\, \hat{u}(k_x)$$

stable equation for integration in positive z, then we must be able to show that its solutions cannot grow in an unbounded way. For an equation in which the total energy does not increase, we would like to be able to write an inequality like[1]

$$||\hat{u}(z,k_x)||_{L_2(z)} \leq ||\hat{u}(0,k_x)||_{L_2(z)} \tag{5}$$

We can see by inspecting Equation (3) that such a bound can be made only for certain values of $\hat{A}(k_x,z)$. Starting with Equation (3) we have

$$||\hat{u}(z,k_x)|| = ||\exp\left[-\int_0^z \hat{A}(k_x,z)\, dz\right] \hat{u}_0(k_x)||$$

$$\leq \exp\left[-\int_0^z \hat{A}(k_x,z)\, dz\right] ||\hat{u}_0(k_x)||$$

$$\leq \exp\left[-z \min_{0\leq z'\leq z} \hat{A}(k_x,z')\right] ||\hat{u}_0(k_x)||$$

Since z is positive and k_x can take on any value, then if

$$\lim_{|k_x|\to\infty} |\mathrm{Re}\ \hat{A}| = \infty$$

which is typically the case[2], the inequality of Equation (5) will only hold if $\mathrm{Re}\ [\hat{A}(k_x,z)] \geq 0$, for all values of $z \geq 0$ and k_x. If Equation (5) holds, then by Parseval's relation we have also that

$$||u(x,z)||_{L_2(x,z)} \leq ||u_0(x)||_{L_2(x,z)} \tag{6}$$

and so Equation (1) will be a stable equation for integration in the positive z sense as well.

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[1] The L_2-norms used in this paper are defined by

$$||u(z)||_{L_2(z)} = \left\{\int_{-\infty}^{\infty} [u(z)]^2\, dz\right\}^{1/2}$$

and

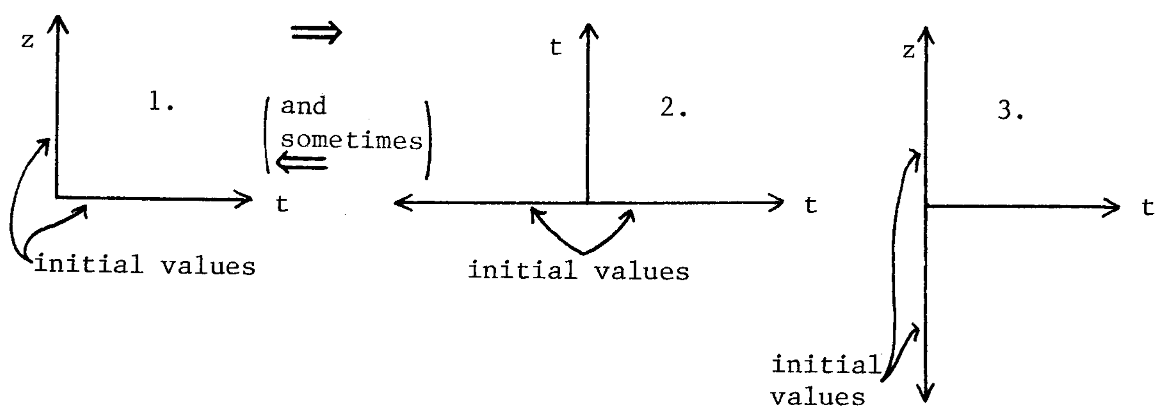
$$||u(x,z)||_{L_2(x,z)} = \left\{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz[u(x,z)]^2\right\}^{1/2}$$

[2] A typical example is the heat-flow operator, $\hat{A} = a(z)k_x^{\ 2}$. Clearly as k_x goes to ∞, so does $\hat{A}$.

To summarize, we have seen here that the condition for stability of Equation (1) is that $\hat{A}$ must be a *positive real* (or zero) *function* for all possible values of z and k_x.

The kind of initial-value problems we deal with in reflection seismology are typically of a more general nature than the problem we stated for Equation (1). Those readers who have programmed a time-domain migration algorithm will recall that the CDP stack or zero-offset section to be migrated is used as the initial data for the z-direction but that initial (or "final") values in time are also needed for the calculation (these are typically zero-values -- see Claerbout, 1976, Equation 11-2-6). So we are interested in problems in which we specify initial values in two dimensions, for instance z and t, instead of just one dimension.

In an article in SEP-8 (p. 48), Björn Engquist argues that in order for a differential equation to be stably integrable in two directions at the same time, a necessary precondition is that it be integrable in either one of the two directions separately, the other direction being replaced by a boundary-value type direction. For some special classes of differential equations this condition is not just necessary but sufficient as well (see Kreiss, 1970).



We have illustrated what we mean in the diagram above. To check the stability of Problem 1, we check the stability of Problems 2 and 3 separately. As pointed out above, this is only a necessary condition for stability, but we believe that for the one-way equations we deal with in migration, it is also a sufficient condition.

Let us consider a simple example of a problem with initial conditions in two directions to see how the stability analysis goes. A dependable old workhorse is the 15-degree equation:

$$(\partial_{zt} + \frac{1}{v}\partial_{tt} - \frac{v}{2}\partial_{xx})\, u = 0 \qquad (7)$$

with initial conditions

$$u(x,t,z=0) = u(x,t)$$
$$u(x,t=0,z) = f(x,z)$$

and $$u_t(x,t=0,z) = g(x,z)$$

We will assume furthermore that there are no boundaries in x. The problem we have just formulated is like Problem 1 in the diagram. We will now formulate and check Problems 2 and 3 for stability.

2. same equation
   initial conditions $u(x,t,z=0) = u_0$
   boundaries: $-\infty < t < \infty$, $-\infty < x < \infty$

3. same equation
   initial conditions: $u(x,t=0,z) = f$
   $\frac{\partial u}{\partial t}(x,t=0,z) = g$
   boundaries: $-\infty < z < \infty$, $-\infty < x < \infty$

Consider first Problem 2. Since there are no boundaries in t or x we can Fourier transform Equation (7) over both these variables to get

$$(-i\,\omega\,\partial_z - \frac{\omega^2}{v} + \frac{v}{2}k_x^{\,2})\,\hat{u} = 0$$

The solution is

$$\hat{u}(z) = e^{-\hat{A}(\omega,k_x)z}\,\hat{u}_0$$

where

$$\hat{A}(\omega,k_x) = -i\left(\frac{\omega}{v} - \frac{vk_x^{\,2}}{2\omega}\right)$$

The function $\hat{A}$ passes the positive real test since its real part is identically zero.

Now consider Problem 3. There are no boundaries in z or x for this problem so we can Fourier transform over these two variables to get

$$(\partial_{tt} + i\, v\, k_z\, \partial_t + \frac{v^2}{2} k_x^{\,2})\, u = 0 \tag{8}$$

This is not in a form which looks like Equation (1), but it still has exponential type solutions,

$$\hat{u}\,(t,k_x,k_z) = \sum_{j=1}^{2} A_j\, e^{-s_j t} \tag{9}$$

where the coefficients A_j depend on the initial conditions $u = f$ and $u_t = g$. For integration in positive t, the positive real test says that the real parts of both s_1 and s_2 must be non-negative for stability. Substituting $A\, e^{-st}$ into Equation (8) we get an algebraic equation whose roots are s_1 and s_2:

$$s^2 - i\, v\, k_z\, s + \frac{v^2}{2} k_x^{\,2} = 0 \tag{10}$$

We get

$$s_{1,2} = \frac{iv}{2}\left(1 \pm \sqrt{k_x^{\,2} + \frac{k_x^{\,2}}{2}}\right)$$

and so the real part of both s_1 and s_2 is zero. Hence problem 3 also passes the positive real test.

Francis Muir has suggested calling operators which pass this double positive real test *causal positive real* (CPR) *operators*.[1] They are called positive real because when we write a differential equation of the form

$$[\partial_z + A(\partial_x, \partial_t)]\, u = 0 \tag{11}$$

the Fourier transform of A over x and t must have a positive real part. They are called causal because all the time-derivatives in Equation (11) must be realized in a causal fashion in order for the equation to be stable in t. Claerbout (1976) talks about causal *vs.* anti-causal realizations of

[1] In accordance with the section of *Fundamentals of Geophysical Data Processing* (Claerbout, 1976) that begins on p. 32, we might also call these operators *generalized impedance functions*.

$(\partial/\partial t)^{-1}$ on page 47 of *Fundamentals of Geophysical Data Processing*. There we see that the causality condition is equivalent to requiring that the Laplace transform of $\partial/\partial t$ have a positive real part, *i.e.* $\partial_t \to -i\omega + |\varepsilon|$ rather than $\partial_t \to -i\omega - |\varepsilon|$. (In the arguments about stability for Problem 3 above, s is actually the Laplace transform of $\partial/\partial t$.) We will therefore call ∂_t a a causal operator if its inverse is causal.

We will close this section with a suggested definition for a CPR operator: Let $I(\partial_t)$ be a differential operator. Then $I(\partial_t)$ is CPR if its Fourier transform $\hat{I}(i\omega)$ satisfies Re $\hat{I}(i\omega) \geq 0$ and if solutions to the equation $[\partial_z + I(\partial_t)]\, u = 0$ depend only on past values of u. It should be emphasized that this is not the only possible definition for a CPR and we may later discover a better one.

2. *The rules of combination for CPR's*

Perhaps the most useful property of CPR operators is that they can be combined in several different ways and still preserve their causal positive real property. We give three of the most useful combinatory rules below along with a sketch of the proofs.

I) $I_1 + I_2 = I_3$ — *Addition*: If I_1 is CPR and I_2 is CPR, then their sum, $I_1 + I_2$, is also CPR. This follows from the fact that the sum of two positive numbers is positive, and that the sum of two causal operators is also causal.

II) $(I_1)^{-1} = I_2$ — *Inversion*: If I_1 is CPR, then its inverse, $(I_1)^{-1}$ is also CPR. "PR" follows from the fact that if a complex number has a positive real part, then its inverse also has a positive real part. The proof for causality is a bit more subtle and for now will be left to the reader. Francis Muir has proved this for the discrete case.

III) $aI_1 + b = I_2$ if $a,b \geq 0$ — *Addition or multiplication with a non-negative real constant*: If I_1 is CPR and a and b are two non-negative real constants, then $aI_1 + b$ is also CPR. This follows because multiplication and addition of a complex number with positive real part with a positive constant will not affect the sign of the result. The causality will not be affected since neither a nor b depend on $\partial/\partial t$.

3. *An example*

In SEP-8 (p. 54) Engquist gives Muir's continued fraction expansion for

$$\sqrt{\frac{1}{v^2}\partial_t^{\,2} + k_x^{\,2}}$$

In slightly different notation it can be written

$$S_{j+1}(\partial_t) = \frac{1}{v}\partial_t + \frac{k_x^{\,2}}{\frac{1}{v}\partial_t + S_j(\partial_t)}$$

where $S_1(\partial_t) = (1/v)\partial_t$, and the $S_j \cong [(1/v^2)(\partial_t^{\,2}) + k_x^{\,2}]^{1/2}$ are progressively higher-order representatives of the square root. It is easy to see that all of the S_j are CPR's if ∂_t is realized in a causal way. The first approximation, $S_1 = (1/v)\partial_t$ is CPR by Rule III, above, since $v \geq 0$. The expression $(1/v)\partial_t + S_1$ is CPR by Rule I. Its inverse, $[(1/v)\partial_t + S_1]^{-1}$, is CPR by Rule II. By Rule III again, $k_x^{\,2}[(1/v)\partial_t + S_1]^{-1}$ is CPR since $k_x^{\,2} \geq 0$. Finally, by Rule I, $S_2 = (1/v)\partial_t + k_x^{\,2}[(1/v)\partial_t + S_1]^{-1}$ is CPR also. Clearly we can replace S_1 by S_j and S_2 by S_{j+1} in the above argument, and by induction, all of the S_j are CPR.

An interesting fact to note is that while all rational expansions of $[(1/v)\partial_t^{\,2} + k_x^{\,2}]^{1/2}$ are CPR, the square root itself is not. In the evanescent region, when $|\omega/v| < |k_x|$, the square root Fourier transforms to $(\omega/v)(v^2 k_x^{\,2}/\omega^2 - 1)^{1/2}$ which can take on either positive or negative real values, and so it is not CPR. Another way of expressing this is that the rational fraction expansion converges to the square root only in the propagating region. In the evanescent region it converges to some harmless CPR operator that does not affect the stability of the differential equations.

4. Stability of differential equations

Consider again an initial-value problem for the differential equation

$$[\partial_z + A(\partial_t)]\, u = 0 \tag{12}$$

Now suppose $A(\partial_t)$ is a CPR operator of order m (i.e. it contains no higher derivatives than $\partial^m/\partial t^m$). Then we would like to be able to say that (12) is a stable equation when we integrate in the positive z sense. Suppose we use initial conditions

$$u(z=0,t) = u_0(t), \quad u_0(t) = 0 \quad \text{for } t \le 0$$

and

$$\frac{\partial^n u}{\partial t^n}(z,t=0) = 0 \qquad n = 0,1,\ldots,m-1$$

We can solve this problem using Laplace transform methods. We define the (two-sided) forward Laplace transform of u(t) by

$$\tilde{u}(s) = \int_{-\infty}^{\infty} dt\, e^{-st}\, u(t) \tag{13}$$

and the inverse Laplace transform by

$$u(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} ds\, e^{st}\, \tilde{u}(s)$$

The integral in Equation (13) will only converge for certain values of Re s. If u(t) is a two-sided function of t, and if u(t) is an L_2 function[1], we can only be certain that the integral will converge if Re s = 0. However, if u(t) is one-sided, with u(t) = 0 for t < 0, then u(t) = H(t)u(t), and Equation (13) becomes

$$\tilde{u}(s) = \int_{-\infty}^{\infty} H(t)\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} u(t)\, e^{-st}\, dt$$

[1] u(t) is called an L_2 function, if $||u(t)||_{L_2} = \left\{\int_{-\infty}^{\infty} [u(t)]^2 dt\right\}^{\frac{1}{2}} < \infty$

which will converge for all values of s whose real part is non-negative, if $u(t)$ is in L_2. We now return to the problem for Equation (12). It is an initial-value problem in t as well as z so we can presume that $u(t) = H(t)u(t)$. If we Laplace transform Equation (12)[1], we get

$$[\partial_z + A(s)]\,\tilde{u} = 0 \tag{15}$$

since $\partial_t \to s$ and $u \to \tilde{u}$. Using the arguments above, this equation is valid for all s with $\mathrm{Re}\, s > 0$ since we cannot write a convergent expression for $\tilde{u}$ otherwise. The solution of Equation (15) can be written in terms of the Laplace transform of the initial conditions, $\tilde{u}_0(s)$:

$$\tilde{u}(s,z) = e^{-A(s)z}\,\tilde{u}_0(s) \tag{16}$$

We recover the time-domain solution by inverse Laplace transforming this result to obtain

$$u(t,z) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} ds\; e^{st}\, e^{-A(s)z}\, \tilde{u}_0(s) \tag{17}$$

If the integral in Equation (17) converges, this will indicate that the original equation, (14), is stable for integration in the positive z direction. To check the convergence, we will make a change of variables. Let $s = -i\omega + \alpha$. Then Equation (17) becomes

$$u(t,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; e^{i\omega t}\, e^{-\alpha t}\, e^{-i[\mathrm{Im}A(s)]z}\, e^{-[\mathrm{Re}A(s)]z}\, \tilde{u}_0(-i\omega + \alpha) \tag{18}$$

Several conditions will have to be met if this integral is to converge. First of all we need

$$\int_{-\infty}^{\infty} d\omega\; \tilde{u}_0(-i\omega + \alpha) < \infty$$

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[1]The result of the Laplace transformation, Equation (15), depends on the fact that we have chosen zero initial values in time. Non-zero initial values would give additional forcing terms in Equation (15) and complicate the proof. We have not looked at this more general problem yet.

A necessary precondition for this is that $\tilde{u}_0(-i\omega + \alpha) < \infty$. For the problem we are considering, this will be true if $\alpha = \text{Re } s > 0$ for the reasons given above. This condition, which is a causality condition, also will assure that the term $e^{-\alpha t} < \infty$. The only other worrisome term is $e^{-[\text{Re } A(s)]\, z}$. For the integral to converge we need $e^{-[\text{Re } A(s)]z} < \infty$ for all possible values of s. This will be guaranteed if $\text{Re } A(s) \geq 0$. Hence, if $A(s)$ is a CPR operator, the integral in Equation (18), and therefore in Equation (17), will converge, implying that Equation (12) is stable for integration in the positive z and t directions.

The end result of this section is that we believe that a sufficient condition for Equation (12) to be stable for integration in the positive t and z sense is that $A(\partial_t)$ be a CPR operator.

5. *Applications*

In Section 3 we showed that the continued fraction expansion of

$$\sqrt{\frac{1}{v^2}\partial_t^{\,2} + k_x^{\,2}}$$

always gives an operator which is CPR. The one-way downgoing wave equation of j-th order can be written

$$[\partial_z + S_j\,(\partial_t)]\, u = 0$$

Since $S_j(\partial_t)$ is a CPR operator for any j, this means that all one-way wave equations derived by using the continued fraction expansions will be stable. This, of course, is in agreement with the results of Engquist who, in SEP-8 (p. 48ff), went through considerably more complicated arguments to make this stability proof.

REFERENCES

CLAERBOUT, J. F. (1976), *Fundamentals of Geophysical Data Processing* (New York: McGraw-Hill).

KREISS, H.-O. (1970), "Initial boundary value problems for hyperbolic systems," *Communications on Pure and Applied Mathematics*, (23), 277-298.