SNELL WAVES

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Reflection seismologists are all familiar with hyperbolic moveout correction. It is the geometric correction which flattens the reflection times to horizontal bedding, provided of course that the earth has a constant, known velocity v. An equation for the correction is

$$t' - t = \frac{(z^2+x^2)^{1/2}}{v} - \frac{z}{v} \tag{0a}$$

which will be compared to the less familiar equation for linear moveout correction

$$t' - t = -px \qquad (p = \text{const}) \tag{0b}$$

While hyperbolic moveout may be able to flatten arrival times over a large portion of a reflection event, with linear moveout only a small portion, called a *patch* of the event, will appear flat. The major advantage of linearly moved out data, as we will see, is that the patches are far more submissive to further analysis than are the hyperbolically moved out data. The patches lead to equations for interval velocity determination which are more accurate and often more sensitive than conventional analysis. The patches from multiple reflections, even though they do not usually come from zero offset, satisfy the usual normal incidence relations exactly, thereby making multiple suppression a more manageable task. It further turns out that a filter-like process known as *slant stack* will enhance the patches for any particular p value and suppress the rest of the data. When slant stacks are made for many p values then all the information of the original data is present and the data may be reconstructed from the slant stacks. Finally, slant stacks simulate waves in the earth. These waves, called *Snell waves*, are plane waves in the special case where v(z) is constant. Although we will not go into the details here, it turns out that Snell waves have an exact migration theory. Such awkward questions as whether to migrate with the stacking velocity, or with the earth velocity, never arise.

One-dimensionality of stratified media

Snell's law says that when a ray crosses or is reflected at a planar contact between two materials, the ratio p, equal to the sine of the angle from ray to perpendicular divided by the material velocity, will be the same after as before the ray intercepted the contact. Snell's law is so basic that it applies in elasticity when a compressional wave is converted to a shear wave. Our discussion will specialize in stratified media, that is, where the velocity v(z) is a function of depth only. Consequently, Snell's parameter

$$p = \frac{\sin\theta(z)}{v(z)} \tag{1}$$

is a constant function of depth. For a ray traveling from a source to a receiver the Snell parameter p is a constant function of time, even if some legs of the journey are by shear waves. Being constrained to make our measurements at the surface of the earth, we cannot make any direct observation of either the material velocity v(z) or the propagation angle θ but the ratio (1) will be easily observed. Figure 1 shows that Snell's parameter p is the inverse of the speed at which the intercept of a wavefront with the earth's surface moves in the horizontal direction. That is,

$$p = \frac{\sin\theta}{v} = \frac{dt}{dx} = \frac{1}{\text{horiz. speed at } z=0} \tag{2}$$

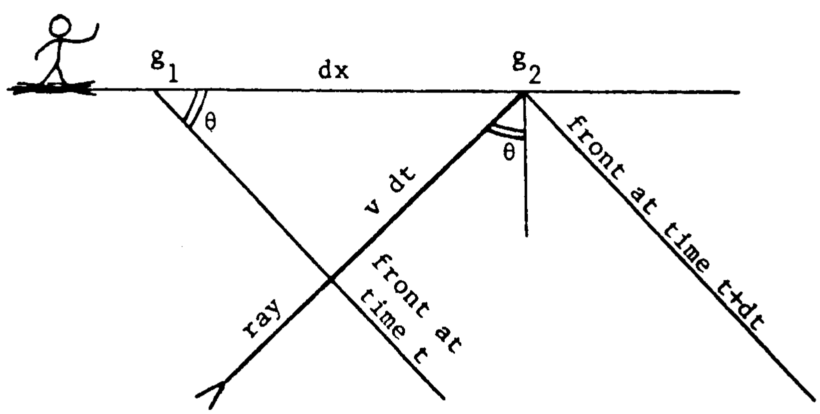


FIGURE 1.--Plane wave arrival at earth's surface showing that observation of dt/dx gives Snell's parameter $p = (\sin\theta)/v$.

The inverse to Snell's parameter p is known as the horizontal phase velocity. For a vertically incident plane wave this velocity is infinity. As long as the phase velocity exceeds the material velocity we are discussing waves. When the phase velocity is less than the material velocity at the surface the disturbances damp out exponentially away from the surface, and the physical behavior becomes quasi-static deformation. The intermediate case, surface waves or ground roll, is for phase velocities in between those of surface material and faster material at depth.

A point source will generate waves in all directions; hence, a wide continuum of Snell parameters. However, by the time the waves get to a distant receiver they may appear to be more like planar waves, that is, more like waves with a fixed numerical value p_0. To analyze these received waves we really do not need all the waves which the source generated at other values of p. To simplify analysis we might ask what sort of source arrangement would generate only waves of one particular numerical value of p. What is needed is a continuously active point source which moves horizontally from $x = -\infty$ to $x = +\infty$ at a speed of $1/p$ [actually, for two-dimensional (x,z) analysis, it would be a line source along the third dimension y]. In a constant velocity medium the waves emitted from this source are plane waves with an angle from the vertical given by $\sin\theta = pv$. In a stratified medium $v(z)$ the wavefronts are no longer planar. Such wavefronts are so central to applied seismogram analysis in petroleum prospecting that they require a name. To prevent us from inaccurately referring to these wavefronts as non-vertically-incident plane waves, I propose to call them Snell waves.

Take a surface Snell wave source to have a horizontal phase velocity p_0^{-1}. It is easily seen from geometry that the wave disturbance as seen at any depth z_0 also moves horizontally at the same speed. Thus, Snell's law (2) is merely a geometrical consequence of the fact that the horizontal phase velocity at any one depth must, for stratified media $v(z)$, equal that at all other depths.

The nice thing about a source of vertically incident plane waves $(p = 0)$ in a horizontally stratified medium is that the ensuing wave field will be spatially one-dimensional. In other words, an observation or a theory for a wave field would be of the form $P(z,t) \cdot \text{const}(x)$. What is true, but not quite so obvious, is that Snell waves for any particular non-zero p value are also spatially one-dimensional. That is, with

$$t' = t - px \tag{3a}$$

$$x' = x \tag{3b}$$

$$z' = z \tag{3c}$$

spatial one-dimensionality is given by the statement,

$$P(x,z,t) = P'(z',t')\cdot\text{const}(x') \tag{4}$$

Obviously when an apparently two-dimensional problem can be reduced to one dimension great conceptual advantages result, to say nothing of computational economic advantages. Before proceeding, study Equation (4) until you realize why the wavefield can vary with x but be a constant function of x' when (3b) says $x = x'$.

Equations (3a,b,c) are a coordinate transformation from (x,z,t) space to (x',z',t') space. Equation 3a is simply a definition of linear moveout. Other papers by this author have considered more complication coordinate transformations. (The spatial coordinates could follow the path of a ray and move at the speed of a front.) In these more advanced papers the readers are asked to delve into such arcane matters as how to manipulate Fourier transforms in the (x',z',t') Snell coordinates and how to express the wave equation and solve it by finite differences in (x',z',t') coordinates. In order to motivate study of these complicated matters, this paper will establish two messages:

Velocity estimation message. The wide offset traces are most sensitive to velocity and contain the most valuable information for velocity determination. A method of analyzing waves in the vicinity of a given Snell parameter $p \pm dp$ provides a simple and accurate means of velocity analysis. In contrast, the familiar power series for traveltime in terms of powers of offset is completely accurate only at small offsets where it is least sensitive to velocity!

Multiple reflection message. Simulation of Snell waves (via *slant stack*, a method yet to be described) is the only way to use reflection seismic data in which the familiar vertical incidence timing and amplitude relations apply to non-zero offset data. Common midpoint stack adds non-zero offset data into zero offset, but because $v(z) \neq$ const, it does so imperfectly and destroys the familiar, normal-incidence timing and amplitude relationships on the multiple reflections. After a half wave-length of timing error, predictive multiple suppression has little value. We will return to this.

Exact graphical method for interval velocity measurement

Consider a point source. The wavefront after a time t is a circle of radius vt and is given by

$$v^2 t^2 = x^2 + z^2$$

Letting f denote the lateral source-receiver offset and z_s denote the depth to an image source under a horizontal plane layer we have

$$v^2 t^2 = f^2 + (z - z_s)^2 \qquad (5)$$

We make our measurements at the earth's surface where $z = 0$. Differentiating (5) with respect to t we obtain

$$v^2 2t = 2f \frac{df}{dt}$$

$$v = \frac{f}{t}\frac{df}{dt} = \frac{f}{pt} \qquad (6)$$

Figure 2 shows that the three parameters required by (6) to compute the material velocity are readily measured on a common midpoint gather.

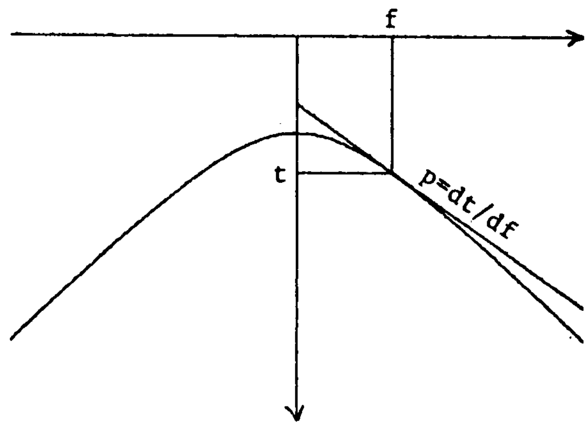


FIGURE 2.--A straight line, drawn tangent to hyperbolic observations. The slope p of the line is arbitrary and it may be chosen so that the tangency occurs at a place of good signal-to-noise ratio.

Of course, we can measure some kind of velocity by means of Equation (6) even if the earth does not have the assumed constant velocity. The question then becomes, what does the measurement mean? In the case of a stratified medium $v(z)$ we can quickly establish the answer to be the familiar RMS, or root-mean-square velocity. To do so, first note that the bit of energy arriving at the point of tangency has throughout its entire trip into the earth been propagating with a constant Snell's parameter p. The best way to specify velocity in a stratified earth is to give it as some function $v(z)$. Another way is to pick a Snell's parameter p and start descending into the earth on a ray with this p. As the ray goes into the earth from the surface $z = 0$ at $t = 0$, the ray would be moving with a speed of, say, $v'(p,t)$. It is an elementary exercise to compute $v'(p,t)$ from $v(z)$ and vice versa. So, when convenient, we may refer to the velocity as some function $v'(p,t)$. The horizontal distance f which a ray will travel in time t is given by the time integral of the horizontal component of velocity, namely

$$f = \int_0^t v'(p,t) \sin\theta \, dt \qquad (7)$$

Replacing $\sin\theta$ by pv from (1) and taking the constant p out of the integral yields

$$f = p \int_0^t v^2 \, dt \tag{8}$$

Inserting (8) into (6) we get

$$v_{measured}^2 = \frac{f}{pt} = \frac{1}{t} \int_0^t v^2 \, dt \tag{9}$$

which justifies the assertion that

$$v_{measured} = v_{root\text{-}mean\text{-}square} = v_{RMS} \tag{10}$$

Equation (9) is exact. It does not involve a "small offset" assumption or a "straight ray" assumption.

Next let us consider the so-called *interval* velocity. Figure 3 shows hyperboloidal arrivals from two flat layers where a straight line of slope p has been constructed tangent to each of the hyperboloids. Both straight lines are constructed to have the same slope p. Then the tangencies are measured to have locations (f_1,t_1) and (f_2,t_2). From (8) and (2), using the subscript i to denote the i-th tangency (f_i,t_i), we have

$$f_i \frac{df}{dt} = \int_0^{t_i} v^2 \, dt \tag{11}$$

Assume that the velocity between successive events is a constant $v_{interval}$ and subtract (11) with $i + 1$ from (11) with i to get

$$(f_{i+1} - f_i) \frac{df}{dt} = (t_{i+1} - t_i) \, v_{interval}^2 \tag{12}$$

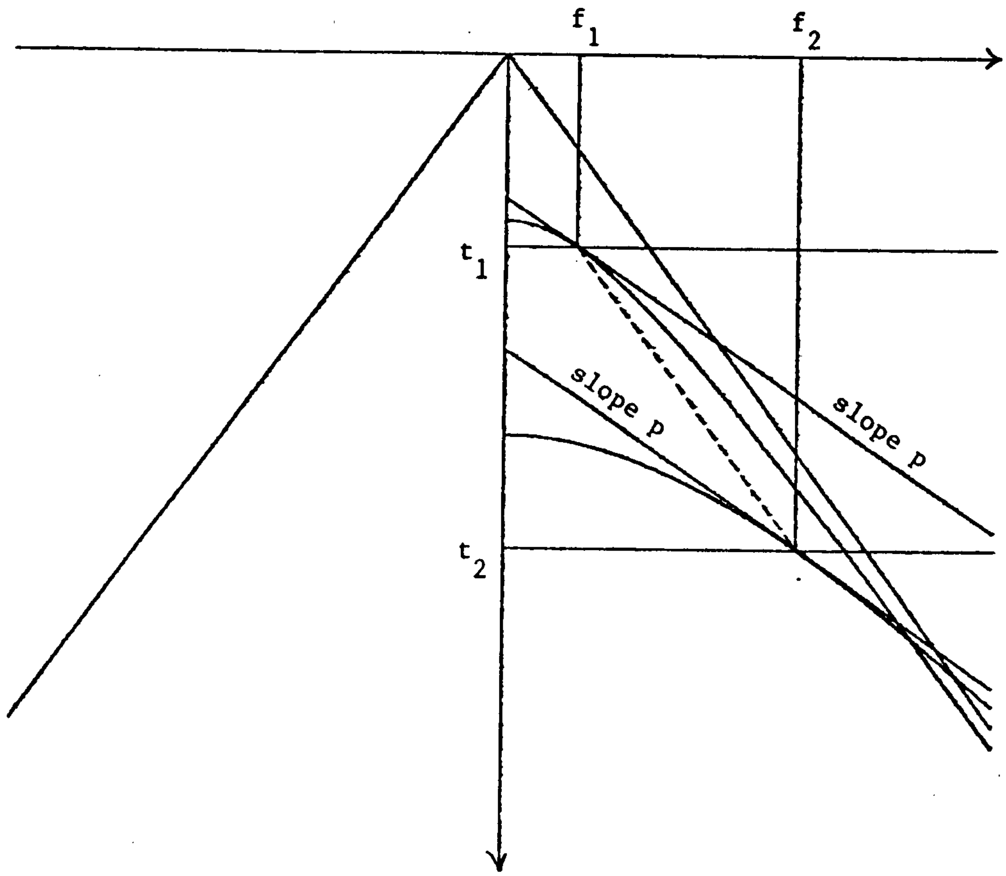


FIGURE 3.--Construction of two parallel lines on a common midpoint gather tangent to reflections from two plane layers.

Solving for the interval velocity,

$$v_{interval}^{2} = \frac{f_{i+1} - f_i}{t_{i+1} - t_i}\frac{df}{dt} \tag{13}$$

So the velocity of the material between the i-th and the i+1-st reflectors can be measured directly by the square root of the product of the two slopes in (13), which are the dashed and solid straight lines in Figure 3. The advantage of manually placing straight lines on the data, over automated analysis, is that you can graphically visualize the sensitivity of the measurement to noise, and you can select the best offsets on the data at which to make the measurement. When doing this routinely one quickly discovers that the major part of the effort is in accurately constructing two lines which are tangent to the events. When this happens, it is convenient to replot the data with linear moveout $t' = t - pf$. After replotting, the sloped lines have become horizontal so that any of the many timing lines can be used. Locating tangencies is now a question of finding the tops of convex events. This is depicted in Figure 4. In terms of the time t', Equation (13) becomes

$$v_{interval}^{2} = \frac{1}{\frac{dt}{df}}\,\frac{1}{p} = \frac{1}{\frac{dt'}{df} + p}\,\frac{1}{p} \tag{14}$$

Finally, the advantages of the manual technique of interval velocity determination presented here, compared with the automated hyperbola scan technique of current industry practice, are:

1) We have made no analytical approximations which deteriorate with angle.

2) We select that portion of the data (by selecting the p value) where the data quality is best for the task at hand.

3) Although it is not shown here, it turns out that migration techniques are available to pre-process the data to remove dip effects.

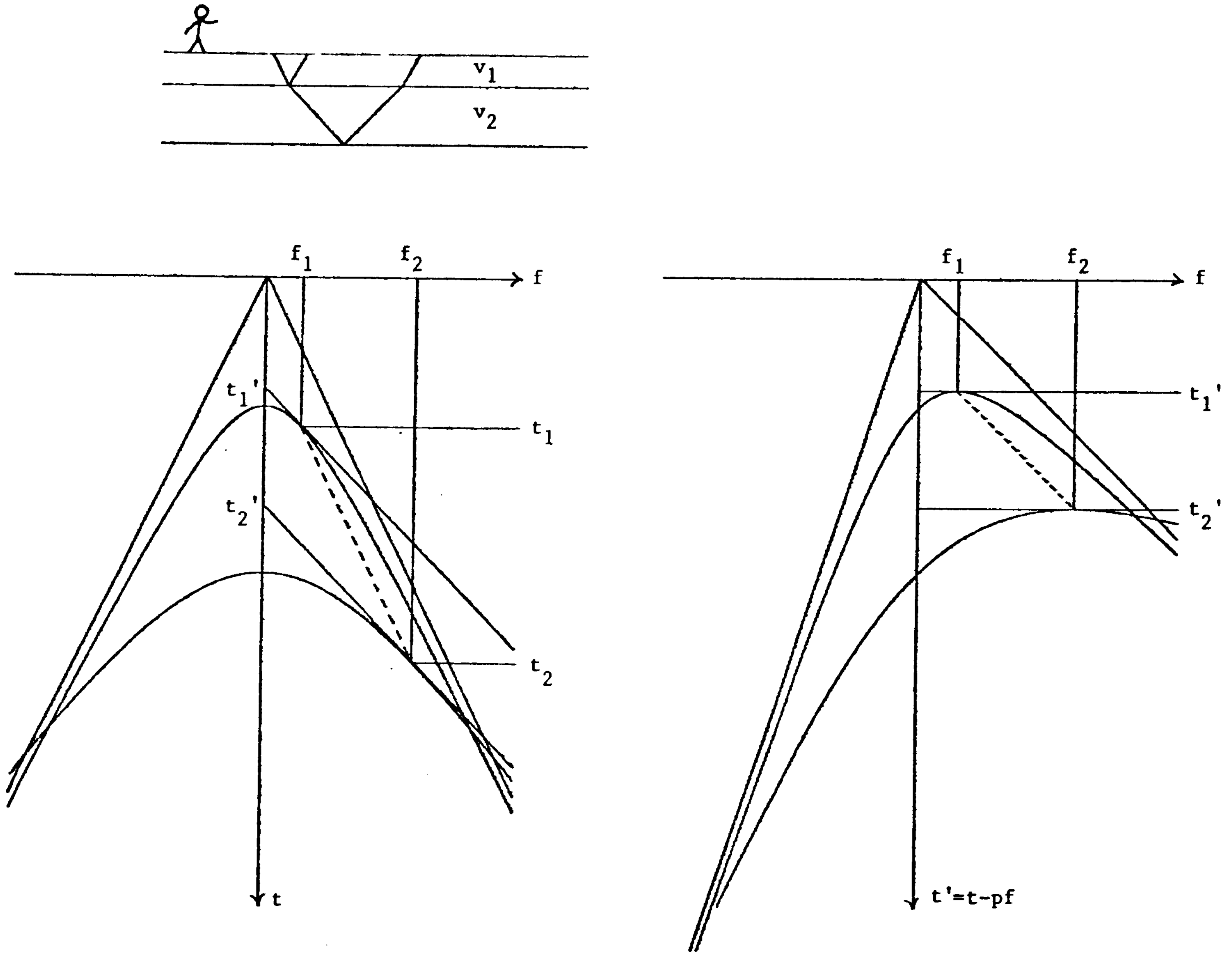


FIGURE 4.--Linear moveout converts the task of identifying tangencies to constructed parallel lines, to the task of locating tops of convex events.

Multiple reflections at non-zero offset

All reflection seismologists are familiar with the timing and amplitude relations of vertical incidence multiple reflections in layered media. To establish this along with some notation let us suppose that we have sea floor two-way traveltime t_1 with reflection coefficient c_1. Then the n-th multiple reflection comes at time nt_1 with reflection strength $c_1{}^n$. Suppose we have also a deeper primary reflection at traveltime depth t_2 with reflection coefficient c_2. Then we expect sea floor peglegs at time $t_2 + nt_1$ with reflection strengths $nc_2c_1{}^n$ (multipled by some transmission coefficients). These familiar normal incidence relationships apply to spherical divergence corrected field data at zero offset, but they do not apply at any other offset. Normal moveout correction would succeed in restoring the timing relationships in a constant velocity earth, so we should ask the questions whether in typical land and marine survey situations $v(z)$ departs so much from constant that residual time shifts greater than a half-wavelength are routinely involved. No equations are needed to get the answer. It is generally observed that conventional common midpoint stacking suppresses multiples because they have lower velocities than primaries. This observation alone implies that normal moveout does indeed often time shift multiples a half-wavelength or more out of the natural zero-offset relationships. As a result, much of the residual multiple reflection energy left in the stack does not fit the familiar vertical incidence model. Consequently, predictive multiple suppression on a common depthpoint stack can be expected to be an exasperating undertaking. You can get rid of the vertically incident energy but the remainder will require least squares coefficients which usually eat up primaries as well as multiples. With marine data the moveout could be done with water velocity, but the peglegs still would not fit the normal incident timing relationship. And the peglegs are often the worst part of the multiple reflection problem.

Advanced research has solved the theoretical problem of prediction and suppression of multiples for both non-zero offset and for irregular non-planar reflectors. The trick for the irregular reflectors uses migration-like techniques which are too complicated to go into here. But the trick for non-vertical incidence is easy and is related to Snell waves. This should not be too surprising when you recall from Equation (4) that a Snell wave source has a response which is a *one-dimensional* function of space.

To see how to relate field data to Snell waves, begin by searching on a common midpoint gather for all those patches of energy (tangency zones) where the hyperboloidal arrivals attain some particular numerical value of slope $p = dt/df$. These patches of energy seen on our surface observations each tell us where and when some ray of Snell's parameter p has hit the surface. Typical geometries and synthetic data are shown in Figures 5 and 6. The traveltimes for all these arrivals satisfy the familiar relationships which we associate with vertical incidence. Three minor differences between this and the vertical incidence case are: 1) the actual numerical values for t_1 and t_2 will change with p because of the different travel path length; 2) the reflection coefficients c_1 and c_2 will change with p because of the different reflection angle; and 3) the non-vertical incidence case theoretically should involve shear waves but for various reasons shear waves are very rarely observed.

Given all the patches of constant p on a gather we can predict the traveltime of multiples by the familiar timing relationships. Unfortunately, the lateral location of any patch depends upon the velocity model $v(z)$. This would seem to imply that you need to do velocity estimation at or before the time that you remove multiples, or that some kind of alternating bootstrap of velocity estimation along with multiple prediction and suppression is required. Luckily, the method of *slant stacking* which is based upon the idea of Snell waves comes to the rescue and enables us to remove multiple reflections before velocity estimation. The procedure of slant stacking is first to do linear moveout with $t' = t - pf$, then to sum over the offset. In other words, you can slant stack in either of two ways: 1) sum along slanted lines in (t,f) space; or 2) do linear moveout $t' = t - pf$ and then sum over offset at constant t'. In either case, the entire gather $P(f,t)$ gets converted to a single trace which is a function of t'. Let us think about what this trace actually is. We will assume that the sum over observed offsets is an adequate representation of integration over all offsets. The (slanted) integral over offset will obviously receive its major contribution from where the path of integration becomes *tangent* to the hyperboloidal arrivals. On the other hand, if rays carry a wavelet with no zero frequency component, and if the arrival time curve crosses the integration curve at any fixed angle, then the contribution to the integral vanishes. Put differently, slant stacking is a sort of narrow band filtering operation which accepts energy at some particular Snell p value and rejects

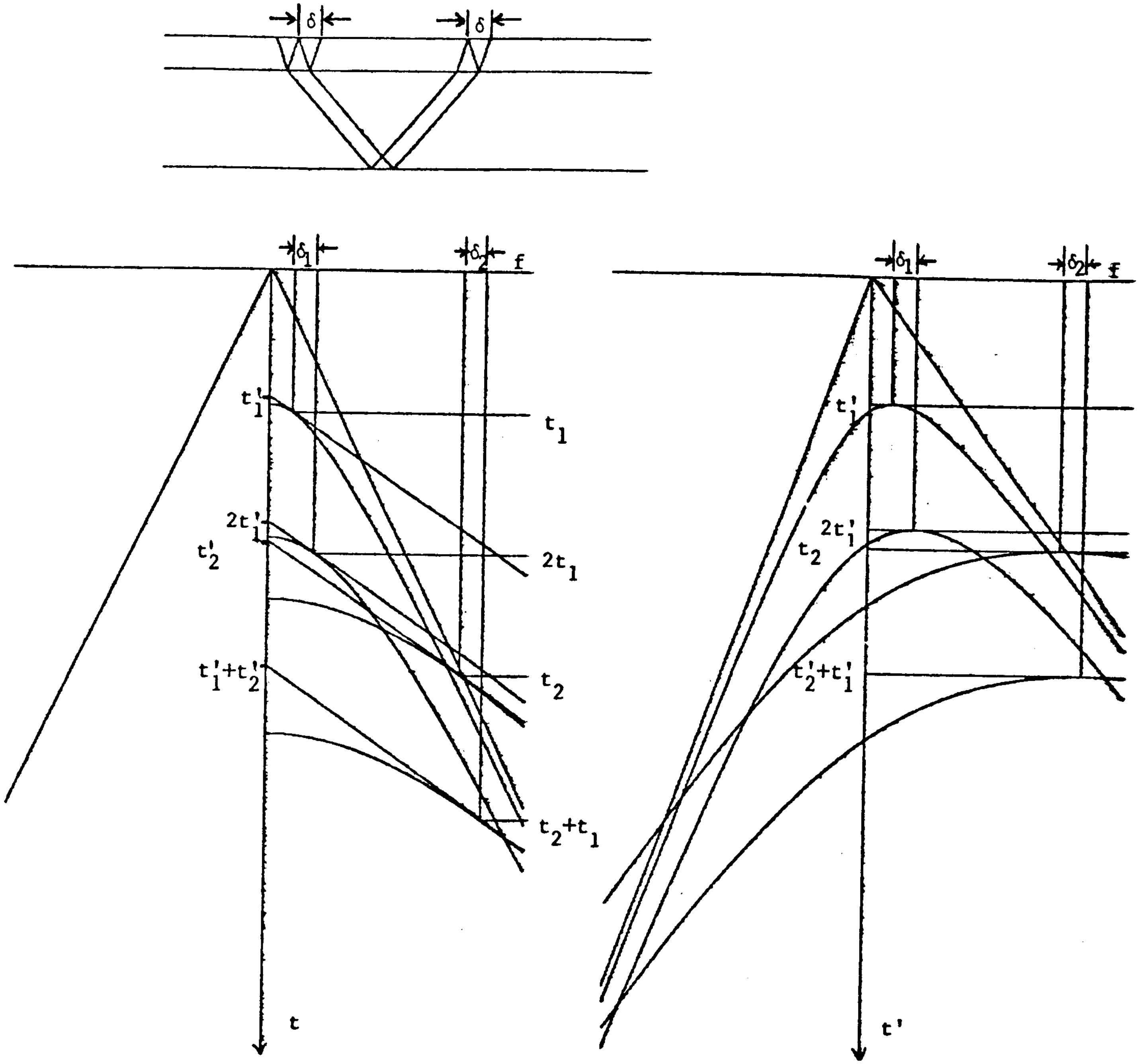
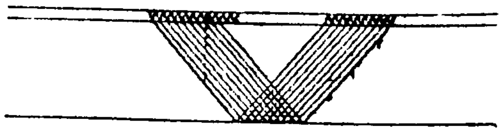


FIGURE 5.—A two layer model showing the events $(t_1, 2t_1, t_2, t_2+t_1)$. Top is a ray trace. On the left is the usual data gather. On the right it is replotted with linear moveout $t' = t - pf$. Plots were calculated with $(v_1, v_2, 1/p)$ in the proportion (1,2,3). Fixing attention on the patches where data is tangent to lines of slope p, we see that arrival times are in the vertical incidence relationships. That is, the reverberation period is fixed, and it is the same for simple multiples as it is for peglegs. This must be so because the ray trace at the top of the figure applies precisely to those patches of the data where $dt/dx = p$. Furthermore, since $\delta_1 = \delta_2$ the times the times $(t_1', 2t_1', t_2', t_2'+t_2')$ also follow the familiar vertical incidence pattern.



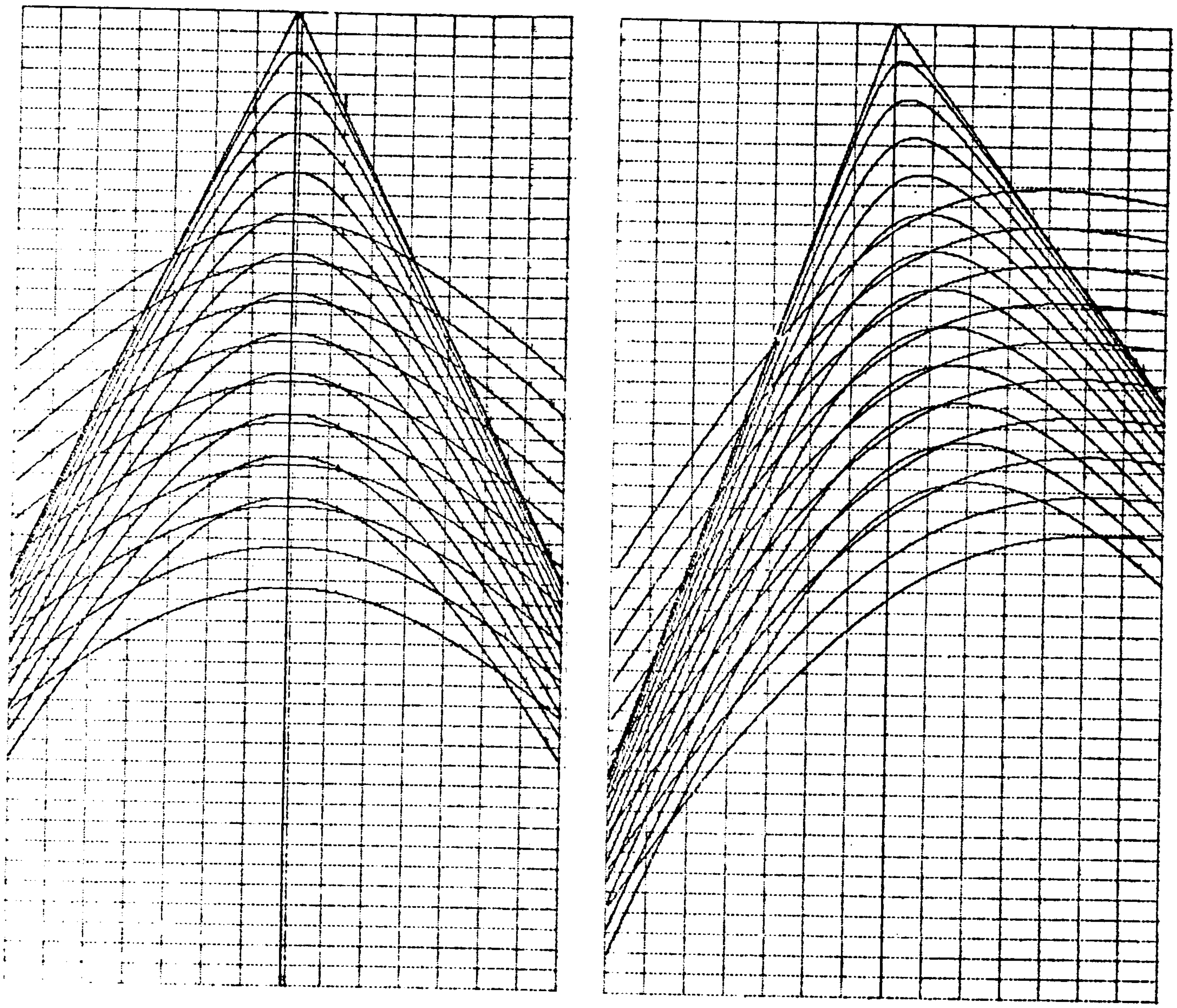


FIGURE 6.--This figure is the same as Figure 5 but more multiple reflections are shown. This simulates much marine data. By picking the tops of all events on the right hand frame and then connecting the picks with dashed lines as in Figure 4, the reader will be able to verify that sea bottom peglegs have the same interval velocity as the simple bottom multiples. The interval velocity of the sediment may be measured from the primaries as in Figure 4. The sediment velocity can also be measured by connecting the n-th simple multiple with the n-th pegleg multiple.

energy at other values. In the frequency domain it is closely related to what is known as *pie slice* filtering. An interesting characteristic of this "filter" is that the gain is proportional to the width of the tangency zone. It may be shown that to a high degree of accuracy this width increases as $t^{1/2}$, which gives half of the spherical divergence correction. In other words, slant stacking takes us from two dimensions to one, but a $t^{1/2}$ remains to correct field data to two dimensions.

Imagine shooting off all shots at the same time to generate a downgoing, normal incident, $p = 0$, plane wave. Such a superposition could be achieved in a computer with conventional split spread data by means of slant stacking at $p = 0$. More general Snell waves could be synthesized by summation at other p values. (Finite difference migration techniques could be used to correctly process these waves where the reflectors are non-flat.)

Of course, we can repeat the slant stacking process for many separate values of p so that the (f,t) space gets mapped into a (p,t) space. It turns out that this mapping is invertible. (The inverse mapping is like the forward mapping followed by frequency domain multiplication by $|\omega|$.) The nice thing about (p,t) space is that the multiple suppression problem decouples into many separate one-dimensional problems, one for each p-value. Not only that, but you do not need to know the material velocity to solve these problems. The one-dimensional inverse problem is a classic one in geophysics with solutions published by many venerable geophysicists. Slant stacked field data along with $t^{1/2}$ cylindrical divergence puts non-zero offset field data into the proper one-dimensional form.

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