MODELING DISPERSION AND ATTENUATION

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Velocity and attenuation characterize the acoustic properties of a material. Velocity is the more important of the two in the acoustic imaging of the earth. Knowledge about attenuation may prove useful in characterizing traps of natural gas or in deconvolving sections which involve unusually attenuating and dispersing geology.

Attenuation is characterized by Q^{-1} , the inverse of the quality factor, which is defined for monochromatic waves as the energy loss per cycle of sinusoidal deformation divided by maximum stored energy. In rocks at seismic frequencies Q is experimentally found to be independent of frequency and amplitude. The last of these implies that any theory of attenuation in earth materials must be linear to be consistent with the data.

We require, therefore, a causal theory in which Q is independent of frequency and in which the wave equation for monochromatic waves is of the same form as the wave equation without attenuation or dispersion.

$$p_{xx} + p_{zz} = [1/v(\omega)^2]\ p_{tt} \qquad (1)$$

A theory which satisfies these conditions is found in the last report of the Stanford Rock Physics project. It begins by noting that since the energy loss per cycle is independent of the period of oscillation, it is reasonable to try a creep function of the form

$$\phi(t) = at^b \qquad (2)$$

The creep function is the strain response to a unit step function stress.

Since the stress-strain relationship is linear, we can write that

$$\Sigma(\omega) = M(\omega)\ E(\omega)$$

$$E(\omega) = S(\omega)\ \Sigma(\omega)$$

$$M(\omega) = 1/S(\omega)$$

where $\Sigma(\omega)$, $E(\omega)$, $S(\omega)$ and $M(\omega)$ are the Fourier transforms of the stress, strain, compliance, and modulus of the acoustic medium. It follows that

$$E(\omega) = i\omega\ S(\omega)\ (1/(i\omega))\ \Sigma(\omega)$$

so by the derivative theorem of the frequency domain. Therefore $i\omega S(\omega)$ is the Fourier transform of the creep function, and that $s(t)$ is the time derivative of the creep function $\phi(t)$. A creep function similar to that hypothesized is used.

$$\phi(t) = H(t)t^{2\nu}/\Gamma\ (1 + 2\nu)K$$

which has time derivative

$$s(t) = H(t)\ 2t^{(2\nu-1)}/\Gamma\ (1 + 2\nu)K \qquad (3)$$

This relation has a Fourier transform

$$S(\omega) = (i\omega)^{-2\nu}/K$$

so that

$$M(\omega) = K(i\omega)^{2\nu} \qquad (4)$$

Equation (4) can be inverse Fourier transformed to give the time domain modulus of the material

$$m(t) = H(t)\ K\ t^{-2\nu}/\Gamma\ (1 - 2\nu) \qquad (5)$$

This is a causal, real function as desired.

We can rewrite (4) as

$$M(\omega) = K\,|\omega|^2 \exp\,(i\pi\nu)\ \mathrm{sgn}(\omega) \tag{6}$$

which implies that the phase difference between stress and strain, $|\delta|$ is $\pi\nu$. If more than one frequency is present we can define

$$1/Q = \tan\,(\delta)$$

so

$$Q^{-1} = \tan\pi\nu\ . \tag{7}$$

We rewrite (7) as

$$\nu = (1/Q)\ \tan^{-1}(1/Q) \cong (1/Q)\ . \tag{8}$$

If we assume that we have a wave equation with phase velocity given by

$$c(\omega) = [m(\omega)/\rho]^{1/2}$$

then its sinusoidal solutions are given by

$$U = \mathrm{Re}\ \exp\,[i(\omega t - k(\omega)z)] = \mathrm{Re}\ \exp\,[i(\omega t - \omega c(\omega)^{-1}z)]$$

$$U = \mathrm{Re}\ \exp\,(-\alpha z)\ \exp\,[i\omega(t - z/c(\omega))]$$

where c is the phase velocity and α is the attenuation. It follows that

$$-\alpha = i^2\omega\ \mathrm{Im}\,\{\,[\rho/M(\omega)\,]^{1/2}\,\}$$

which, using equation (6), gives

$$\alpha = |\omega|^{1-\nu} \sin(\pi\nu/2) \ (\rho/K)^{1/2} \ \mathrm{sgn}(\omega) \tag{10}$$

Similarly, the phase velocity satisfies the relation

$$c = \mathrm{Re}\ [M(\omega)/\rho]^{1/2} = (K/\rho)^{1/2}\ \mathrm{Re}\ [(i\omega)^{2\nu}]^{1/2}$$

$$= (K/\rho)^{1/2}\ \mathrm{Re}\ [\ |\omega|^{2\nu} \exp(i\pi\nu)\ ]^{1/2}$$

$$= (K/\rho)^{1/2}\ |\omega|^{\nu} \cos(\pi\nu/2) \tag{11}$$

We can combine (10) and (11) to get

$$c = c_o\ |\omega|^{\nu}$$

$$= |\omega|^{1-\nu}\ c_o \tan(\pi\nu/2)\ \mathrm{sgn}(\omega)$$

$$c_o = (K/\rho) \quad \cos(\pi\nu/2)\ , \tag{12}$$

Now that we have $\alpha(\omega)$ and $c(\omega)$ we may want to migrate with them included in the wave equation. With finite differences, we can migrate each frequency component separately and sum the results.

We start with the 15 wave equation for upcoming waves

$$p_{xx} + (2/v)p_{zt} = 0 \tag{13}$$

and express $p(x,z{=}0,t) = Q(x,z{=}0,\ ) \exp(i\omega t)$

so that

$$\Sigma\ (Q_{xx} + 2i\omega/v)\ Q_z) \exp(i\omega t) = 0$$

If it is safe to assume nearly vertical propagation we can substitute ωt by $(\omega z/v)$ and then discretize. Letting k denote the x index and j denote the z index, we set

$$\Sigma \; [Q_k^{j+1} + (-2+ia)Q_k^{j+1} + Q_k^{j+1}] \; \exp\,(i\omega z/v)$$

$$= \Sigma \; [Q_{k+1}^{j} + (2+ia)Q_k^{j} - Q_{k-1}^{j}]$$

where

$$a = (4i\omega\Delta z)/(v\Delta x\Delta x) \; .$$

If attenuation is to be included the $\exp\,(i\omega z/v)$ in the above equation needs to be replaced by $\exp\,[i\omega z/v + \alpha(\omega)z]$. In practice this makes the algorithm unstable so an exponential gain has to be applied in the course of migration. The replacement is then with $\exp\,(i\omega z/v + \alpha z - gz)$, where g is a suitably chosen constant or function of z .

The algorithm proceeds by first transposing the unmigrated time section and then taking the Fourier transform of all of the traces. The traces are then time reversed in the frequency domain by taking advantage of the symmetry properties of the Fourier transform. This matrix is then transposed back and the various x strips of different frequency value are then read off and migrated downwards one at a time. These submigrations can be summed and the complex sum inverse Fourier transformed to give a migrated section. Q in this algorithm can be taken as a function of x and z .

It is also possible to model constant Q sediment using the FK scheme. The derivation of the algorithm proceeds as usual except the projection operator is taken to be $\exp\,(ik_z z + \alpha z)$ instead of $\exp\,(ik_z z)$. We apply this operator to the Fourier transform over x and t of the surface data and get

$$p(x,t,z) = \iint P(k,\omega,z=0)\; \exp(ik_z z + \alpha z - i\omega t + ikx)\; d\omega\; dk \qquad (14)$$

We seek $p(x,t=0,z)$ and therefore need to change variables of integration from ω to k_z. This is done using the dispersion relation

$$k_z^2 = \omega^{2(1-\nu)} v_0^{-2} - k^2 . \tag{15}$$

This can be rearranged to give

$$\omega = v_0^{1/(1-\nu)} (k^2 + k_z^2)^{1/2(1-\nu)} \tag{16}$$

and can be differentiated to yield

$$d\omega = v_0^{1/(1-\nu)} (k_z^2 + k^2)^{(2\nu-1)/2(1-\nu)} (1-\nu)^{-1} dk_z . \tag{17}$$

Substituting (15) into the expression for α yields

$$\alpha = (k_z^2 + k^2)^{1/2} \tan(\pi\nu/2) . \tag{18}$$

Between (16), (17), (18), and setting $t=0$ to get the migrated section, we can get an ω-independent integrand. If we approximate

$$\exp(\alpha z) = 1 + \alpha z$$

then the integral (14) becomes for $t=0$

$$p(x,t=0,z) = \iint dk_z\, dk\, P(k_z,k,z=0) \exp(ik_z a + ikx)$$
$$[1 + (k_z^2 + k^2)^{1/2} \tan(\pi\nu/2)]\, v_0^{1/(1-\nu)} (k_z^2 + k^2) \tag{19}$$

where we have used

$$P(k_z,k,z=0) = P(k_z,\omega=v_0^{1/(1-\nu)} (k_z^2 + k^2)^{1/2(1-\nu)},z=0) .$$