STOLT'S STRETCHING FUNCTIONS

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We start with the wave equation in Cartesian coordinates and transform to primed coordinates. The new system will take into account the difference between real and apparent depths caused by the z-dependence of velocity. The new system will also be a generalized version of retarded time coordinates, in which the $P'_{z'z'}$ term is Doppler shifted to zero, and in which the thin lens term is neglectable.

The relevant transformation equations are

$$t' = t + \int_0^z \frac{dz}{v(z)}$$

$$z' = \frac{1}{v_0} \int_0^z v(z)\ dz$$

$$x' = x$$

$$P'(x',z',t') = P(x,z,t)$$

The second of these equations can be explained by building a thin-plate model of the earth. We consider a plate at depth z, below the surface and a point scatterer lying below it at depth z_2+z_1. Assume that the velocity is v_0 everywhere except inside the plate, in which the velocity is $v(z)$. Finally, call the apparent depth of the scatterer $z_2'+z_1$. We have using Snell's Law

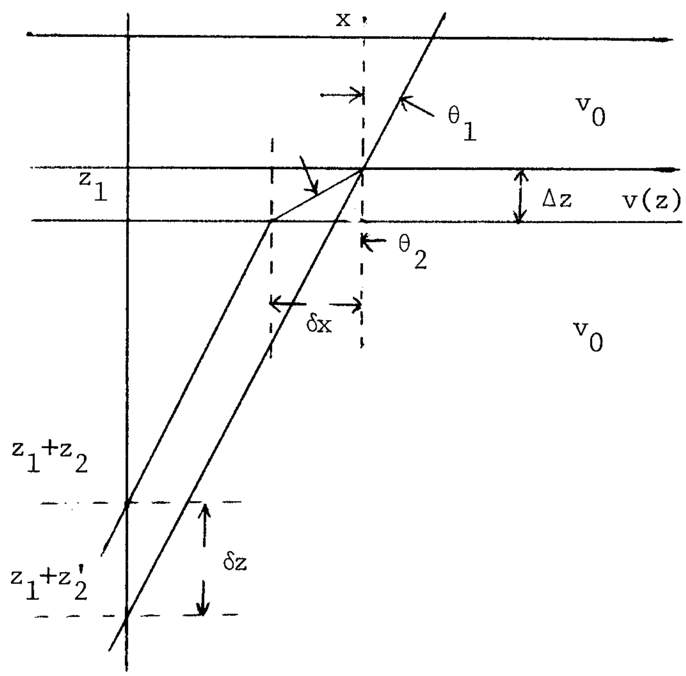
$$\delta x = \Delta z \tan \theta_2$$

$$\approx \Delta z \sin \theta_2$$

$$\delta x \approx \frac{v}{v_0} \Delta z \sin \theta_1$$

$$z_2 \approx (x - \delta x) \frac{1}{\tan \theta_1}$$

$$z_2 \approx \frac{x - \delta x}{\sin \theta_1}$$



$$z_2' = \frac{x}{\sin \theta_1}$$

$$\delta z' = \frac{v}{v_0} \Delta z$$

If we were to integrate these from $z=z'=0$ to z, we would get the total apparent depth of the point in the layered earth. In other words

$$z' = \frac{1}{v_0} \int_0^z v(z)\, dz$$

Note that $v(z) = \frac{dz}{dt}$, so that, for arrivals at time t,

$$z' = \frac{1}{v_0} \int_0^t v^2(t)\, dt = \frac{1}{v_0} v^2(t)_{rms} t \quad .$$

Next, we use this new coordinate system to construct a new version of the wave equation. Unfortunately, we will find that the final result will have a complicated structure - the condition for stopping the downward continuation of the geophones will be messy. Using the chain rule, we obtain

$$P_x = P'_{x'}$$

$$P_z = P'_{z'}Z'_z + P'_{t'}t'_z = \frac{v(z)}{v_0} P'_{z'} + \frac{1}{v(z)} P'_{t'}$$

$$P_t = P'_{t'}t'_t + P'_{z'}z'_t = P'_{t'}$$

The wave equation $P_{xx} + P_{zz} = \frac{1}{v^2} P_{tt}$ becomes, if

$d_z v_z(z) << v_0$,

$$0 = P'_{x'x'} + \left(\frac{v}{v_0}\right)^2 P'_{z'z'} + \frac{1}{v(z)^2} P'_{t't'} + \frac{2}{v_0} P'_{z't'} - \frac{1}{v(z)^2} P'_{t't'}$$

$$0 = P'_{x'x'} + \left(\frac{v}{v_0}\right)^2 P'_{z'z'} + \frac{2}{v_0} P'_{z't'}$$

The migration consists of changing the field $P(x,z=0,t)$ to the final field $P(x,z=\delta(t'), t=0)$ where we have made use of the equation:

$$t' = t + \int_0^z \frac{dz}{v(z)} = \left.\int_0^{\delta(t')} \frac{dz}{v(z)}\right]_{t=0}$$

The stopping point is $z=\delta(t')$. An equivalent stopping point for the apparent depth z' is

$$\left.z'\right]_{t=0} = \frac{1}{v_0}\eta(t') = \left.\frac{1}{v_0}\int_0^z v(z)\, dz\right]_{z=\delta(t')}$$

$$\eta(t') = \int_0^{\delta(t')} v(z)\, dz = v_{rms}^2\, t'$$

Thus in the primed coordinate frame we have a final, and complicated, field of

$$P'\left(x', z' = \frac{1}{v_0}\eta(t'),\quad t' = \int_0^{\delta(t')} \frac{dz}{v(z)}\right)$$

Instead of this, it would be convenient if the final value of the depth coordinate were proportional to the final value of the time coordinate.

We therefore define a new depth coordinate d by the equation

$$d = \frac{z' v_0}{\eta} \; D(t')$$

where $D(t')$ will be the new, and so far undetermined, time coordinate. We will want to choose D so that the new wave equation is nearly independent of velocity. We set

$$P'(x',z',t') = P''(x',d,D) = P''(x,d,D)$$

in which case

$$P'_{z'} = P''_d \, d_{z'} = \frac{v_0}{\eta} D \, P''_d$$

$$P'_{z't'} = P''_{dd} \, d_{z'} d_{t'} + P''_{dD} \, d_{z'} \, D_{t'}$$

$$P'_{z't'} = P''_{dd} \frac{v_0 D}{\eta} \; z' v_0 \frac{d}{dt'}\left(\frac{D}{\eta}\right) + P''_{dD} \frac{v_0 D}{\eta} \; \frac{d}{dt'} (D)$$

The condition that our wave equation stay approximately independent of v is equivalent to saying that the coefficient in front of the P''_{dD} be a constant and that all of the velocity dependence be dumped into the P''_{dd} term. We require that

$$\frac{v_0 D}{\eta} \frac{d}{dt'} (D) = v_0$$

which has a solution

$$D = \sqrt{2 \int_0^{t'} \eta(t')\, dt'}$$

Migration now consists of changing the field at the surface $P''(x,d=0,D)$ into the field at depth when $t=0$

$$P''(x,d=D,D>0)$$

The new version of the wave equation is found by using the chain rule. We have

$$D = \sqrt{2 \int_0^{t'} \eta(t')\, dt'}$$

so that

$$\frac{dD}{dt'} = \frac{\eta}{D}$$

$$\frac{d}{dt'}\left(\frac{D}{\eta}\right) = \frac{1}{D} + D \frac{d}{dt'}\left(\frac{1}{\eta}\right) = \frac{1}{D} - D \frac{v^2}{\eta^2}$$

so

$$P'_{z't'} = P''_{dd} \frac{z' v_0^2}{\eta} - P''_{dd} \frac{z' v^2 D^2 v_0}{\eta^\eta} + v_0 P''_{dD}$$

$$P_{z'z'} = \frac{v_0^2 D^2}{\eta^2} P''_{dd}$$

The wave equation is, dropping primes,

$$0 = P_{xx} + \left[\frac{v_0^2 D^2}{\eta^2} \left(\frac{v^2}{v_0^2} \right) + \left(\frac{2}{v_0} \right) \left(\frac{z' v_0^2}{\eta} - \frac{z' v^2 D^2 v_0}{\eta^\eta} \right) \right] P_{dd} + 2P_{dD}$$

$$0 = P_{xx} + \left[\left(\frac{vD}{\eta} \right)^2 + \frac{2d}{D} \left[1 - \frac{v^2 p^2}{\eta^2} \right] \right] P_{dd} + 2P_{dD}$$

$$0 = P_{xx} + W(x,d,D) P_{dd} + 2P_{dD}$$

Generally the $W\ P_{dd}$ is small, and neglectable if the dips are small, so it is expedient to substitute an average value for $W(x,d,D)$.

The new wave equation has a new dispersion relation. If we start with a two way travel time axis in the unmigrated time section, this dispersion relation is

$$W\ k_d^2 + k^2 - r\omega k_d = 0$$

where we have taken ω, k_d , and k to be the Fourier transform variables of D, d , and x , respectively. Using this relation the projection operator is of the form

$$\exp\ [-i\ (4\ (\omega/W)^2 - k_d^2/W)^{0.5}]$$

The scaling factor in the FK integral also has its factors appropriately scaled by W .

The computational strategy for setting a migrated time section is straightforward. We simply calculate the value of $\eta(t)$ at the surface, where $t=t'$, and plus this function into the integral expression for $D(t')$. This gives us a transformation before migration

$$d = \left[2 \int_0^t v_{rms}^2 (t)\ t\ dt \right]^{1/2}$$

After migration the section is in (x,d) space. But $d=D$ in this section, so we can use the mapping already constructed in the other direction to get the wave field as a function of t' again. Getting depth sections is another matter and involves some vertical ray tracing and axis stretching.

References

[1] Stolt, R. H., "Migration by Fourier Transform," *Geophysics*, 43, (1978), p. 23