A DISCUSSION OF VARIABLE VELOCITY AND FAULTY RAY-TRACING EFFECTS OF ONE-WAY WAVE EQUATIONS

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Introduction

One-way wave equations which model wave propagation in a variable velocity medium have been developed in the last SEP report (see articles by Engquist and Brown, SEP-13). A thorough discussion was not made, however, of the effects which these equations simulate, and this has unfortunately led to some confusion regarding the significance of those effects. In this paper the derivation of these equations is reviewed for the simpler case of a one-dimensional medium, and the effect of the new terms is investigated by looking at general solutions of the equations. It is seen that the first new terms which contain velocity derivatives represent purely amplitude effects: they model the accumulated transmission coefficient effects of the inhomogenous medium, and contain no information about the refraction of the waves. Incorrect refraction of downward-continued waves, which has been a persistent problem associated with the migration process, is shown in the last section of this paper to be an effect caused by the inaccuracies involved in approximating the square-root in the factorization development of the migration equation.

The Variable Velocity Equations

The concept of wave equation factorization has been discussed in recent SEP reports and has been part of the exploration seismology folklore for quite some time. The idea is quite simple: the acoustic wave equation is factored into two differential equations, both of which contain only single z-derivatives and which describe up- and down-going wave motion respectively. For a one-dimensional problem, this factorization is exact when the velocity, v, is constant. The factorization

$$(\partial_z + \frac{1}{v}\partial_t)(\partial_z - \frac{1}{v}\partial_t)P = (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P \tag{1}$$

gives the two equations

$$P_z + \frac{1}{v} P_t = 0 \tag{2a}$$

and $$P_z - \frac{1}{v} P_t = 0 \ . \tag{2b}$$

It is readily verified that the general solutions to these equations are of the form

$$P(z,t) = f(t - \frac{z}{v}) \tag{3a}$$

and $$P(z,t) = f(t + \frac{z}{v}) \tag{3b}$$

respectively. The first equation thus corresponds to the downgoing waves and the second to upcoming waves.

In a two-dimensional medium, the analogous factorization is given by

$$\left[\partial_z + \left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)^{1/2}\right]\left[\partial_z - \left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)^{1/2}\right]P = 0. \tag{4}$$

This factorization is also exact, although the "pseudo-differential" operator indicated by $\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)^{1/2}$ is not realizable in the time domain due to its "non-local" nature. It is only defined through the effect of its Fourier-transform in the frequency domain. As a result, in order for finite difference techniques to be used for solution of the one-way wave equations indicated by the factorization in equation (4), the square root must be approximated in some way. The approximation of this square root has been much discussed in previous SEP reports (see for example articles in SEP-8 by Engquist and Claerbout) and will be discussed also in the last section of this paper. This square-root approximation should not be confused with the asymptotic series approximations which will be discussed in this section.

The factorizations implied by equations (1) and (4) will not give the proper result when expanded out if the medium velocity is allowed to be variable in x and z. In fact there is *no* factorization which will give back the wave equation exactly for the variable velocity case. The physical meaning of this result is simply that there is interaction of a complex nature between the upcoming and downgoing parts of the solution if the velocity varies, for example reflection and transmission effects. It is still possible, however, to *approximately* separate the two types of waves, and it will be seen that the geometrical tracing of rays through the medium and the calculation of simple amplitude effects can be done quite well.

Consider again the factorization of the one-dimensional wave equation (1), but this time allow v to b a function of z.

$$\left(\partial_z + \frac{1}{v(z)}\partial_t\right)\left(\partial_z - \frac{1}{v(z)}\partial_t\right)P = 0 \qquad (5)$$

Expanding this, the following equation for P is obtained, which is clearly <u>not</u> the wave equation:

$$P_{zz} - \frac{1}{v^2}P_{tt} + \frac{v_z}{v^2}P_t = 0 \qquad (6)$$

One might hope to rectify this problem by adding an additional term to one of the factors, but it will be quickly seen that the addition of each new term will result in even more terms, and the difficulties quickly become unmanageable. This suggests that the factorization can only be done up to some error terms, but one might hope that these error terms can be made as small as possible. The problem can be treated in general by proposing a factorization of the form

$$\left[\partial_z + \lambda\ (v,t,\partial_t,1,\partial^t,\ldots)\right] \left[\partial_z - \lambda\ (v,t,\partial_t,1,\partial^t,\ldots)\right] P = 0 \qquad (7)$$

where with gross abuse of mathematical terminology, λ is allowed to be an asymptotic series in ∂^t as indicated below.

$$\lambda = \lambda_1 \partial_t + \lambda_o + \lambda_{-1}\partial^t + \ldots + \lambda_{-j}(\partial^t)^j + \ldots, \qquad (8)$$

with this expansion being valid for "large $|\partial_t|$", i.e. high temporal frequencies.*

The factorization can then be made accurate to within terms of some order of ∂^t by including the appropriate number of terms, i.e.,

$$(\partial_z + \lambda_1\partial_t + \lambda_o + \ldots)\ (\partial_z - \lambda_1\partial_t - \lambda_o - \ldots)P$$

$$= \left(\partial_{zz} - \frac{1}{v^2}\partial_{tt}\right)\ P + 0\left[\left(\partial^t\right)^n P\right]. \qquad (9)$$

The resulting one-way equations will be called the "n-th order equations". The coefficients λ_j can be determined recursively one at a time. The first coefficient λ_1, is found by deriving the "(-1)st order equations".

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* To be mathematically proper these arguments should be carried out in the frequency domain, in which case ∂_t goes to $-i\omega$ and ∂_z goes to ik_z, but this actually just makes for unnecessary complication. The "large $|\partial_t|$" limitation thus becomes a "large ω" or high-frequency approximation. I find it more convenient to think of the "large $|\partial_t|$" idea: If ∂_t is applied to the wavefield P, then if $|\partial_t P|$ is large, that means that the wavefield is changing very rapidly at some point in space-time. This would be true when a wave-front suddenly passes the point of observation, and it is wave fronts one is interested in here.

It is desired that

$$(\partial_z + \lambda_1 \partial_t + \ldots)\ (\partial_z - \lambda_1 \partial_t - \ldots)P$$

$$= (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P \ + \ 0\left[\left(\partial^t\right)^{-1} P\right]$$

$$= (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P \ + \ 0(\partial_t P). \tag{10}$$

Expanding the left-hand side,

$$\partial_{zz} + \lambda_1 \partial_t \partial_z - \lambda_1 \partial_z \partial_t - (\partial_z \lambda_1)\partial_t - \lambda_1^2 \partial_{tt} + \ldots$$

$$= \partial_{zz} - \frac{1}{v^2}\partial_{tt} \ + \ 0(\partial_t). \tag{11}$$

On the left-hand side, the second and third terms cancel, the fourth term is $0(\partial_t)$, and may be ignored, and so this gives that

$$\lambda_1 = \frac{1}{v} . \tag{12}$$

The "(-1)th-order equation" is then given by

$$(\partial_z - \lambda_1 \partial_t)P = 0$$

or $$(\partial_z - \frac{1}{v}\partial_t)P = 0 , \tag{13}$$

which is the same equation derived when velocity effects were excluded. Repeating the process, λ_o is determined:

$$(\partial_z + \lambda_1 \partial_t + \lambda_o)\ (\partial_z - \lambda_1 \partial_t - \lambda_o)P$$

$$= (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P \ + \ 0\left[(\partial^t)^0 \ P\right]$$

$$= (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P \ + \ 0(P). \tag{14}$$

Substituting in for λ_1 and expanding the left-hand side,

$$(\partial_{zz} - \frac{1}{v^2}\partial_{tt} + \frac{v_z}{v^2}\partial_t - (\partial_z\lambda_o) - 2\lambda_1\lambda_o\partial_t + \lambda_o\partial_z - \lambda_o^2)P$$

$$= (\partial_{zz} - \frac{1}{v^2}\partial_{tt})P + O(P) \qquad (15)$$

Dropping terms of order P and lower,

$$\frac{v_z}{v^2}\partial_t = 2\lambda_1\lambda_o\partial_t \qquad (16)$$

So the "0'th-order equation" is given by

$$(\partial_z - \frac{1}{v}\partial_t - \frac{v_z}{2v})P = 0. \qquad (17)$$

This process can be extended further to obtain higher order equations.

It should be noted here that variable velocity equations for a *two*-dimensional medium have been derived already in Engquist ("Variable Velocity: Wave Extrapolation and Reflection", SEP-13) and in Brown ("One-Way Wave Equations by Factorization into Pseudo-Differential Operators for the Variable Coefficient Case", SEP-13). The derivation for the one-dimensional medium has been included here in order to clear up some confusion about the latter article and also so that the effects of these extra terms might be discussed. For the two dimensional case the arguments made in the derivation are essentially the same, although the terms in the asymptotic expansion are allowed to be arbitrary functions of decreasing orders of homogeneity in ∂_t and ∂_x. (See Brown, SEP-13).

To look at the effects of the terms which are added by the asymptotic expansion approach, general solutions of these equations can be considered. The "-1'st-order equation", equation (13), is recognized to be the one-dimensional equivalent of the migration equation in non-time-retarded coordinates. A quick substitution verifies that any function of the form

$$P(x,t) = f(v(z)t+z) \qquad (18)$$

will satisfy this equation. In particular if initial conditions are specified corresponding to a delta-function type impulse at time zero:

$$P(x,t=0) = \delta(z),$$

then the equation will propagate the impulse through the medium with the proper velocity but with none of the amplitude changes which would be expected in a variable-velocity medium due to the continuous back-scattering of energy resulting from inhomogeneities:

$$P(x,t) = \delta(v(z)t + z). \tag{19}$$

It can be shown that including the λ_o term in the differential equation will result in the solutions which mimic the energy loss due to back scattering. The differential equation is

$$(\partial_z - \frac{1}{v}\partial_t - \frac{v_z}{2v})P = 0 \; . \tag{20}$$

which can be solved quite easily for the initial conditions above by changing to retarded time variables and integrating. The solution is

$$P(x,t) = \left(\frac{v(0)}{v(z)}\right)^{1/2} \delta(z + v(z)t) \; . \tag{21}$$

The square-root term is just the accumulated transmission coefficient for the medium. (It is possible to show that this is the proper transmission coefficient by considering the case with a sharp interface in the medium. The reader may verify that using the technique suggested above to get (21), the proper interface transmission coefficient for an acoustic medium will be obtained).

If yet another term of the asymptotic expansion is included, the "1'st-order" one-way wave equation is obtained:

$$P_z - \frac{1}{v}P_t - \frac{v_z}{2v}P - \left(\frac{(v_z)^2}{8v} - \frac{v_{zz}}{4}\right)\partial^t P = 0 \; . \tag{22}$$

It is possible to show that the new term obtained contains purely phase effects, i.e. it is a propagation term and not an amplitude term. (This can be demonstrated by substituting a trial solution of the form

$$\exp\left[i\omega(t - a(z) - \frac{1}{\omega}b(z) - \frac{1}{\omega^2}c(z) - \ldots)\right]$$

into (3-17) and grinding through, for example). The alert reader may have noticed by now that the whole discussion in this section has many similarities to the techniques used in geometrical optics (see for example: Whitham: *Linear and Nonlinear Waves*). The asymptotic series used in the differential equations here correspond to the asymptotic solutions of the wave equation used in geometrical optics theory. This new term in equation (22) corresponds directly to the term in that theory which contains information about the behavior of the wave equation solutions near *caustics*. Caustics are regions in the wave field where the rays predicted by simple geometrical ray tracing begin to cross and where the ray-tracing approach is therefore invalid. Equation (3-9), and its two-dimensional equivalent, the migration equation, are essentially just modelling this geometrical ray tracing, and this indicates that the new term in equation (3-17) is unimportant in regions where simple geometrical ray theory is valid.

To summarize the main points of this section, if there are no regions where geometrical ray-theory would predict ray-crossing, equation (3-9) and its two dimensional equivalent,

$$\left[\partial_z - \left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)^{1/2}\right] P = 0 \quad , \tag{23}$$

(the migration equation without square-root approximation), contain all the necessary terms to propogate waves to the correct location but ignoring the transmission coefficient effects of the medium. If the second asymptotic series term is included, equation (17) or its two dimensional equivalent,

$$\left[\partial_z - \frac{1}{v}\partial_t\left(1 - v^2\partial_{xx}^{tt}\right)^{1/2} - \frac{\frac{v_z}{2v}\left(1 - v^2\partial_{xx}^{tt}\right)^{1/2} + \frac{v_x}{2}\partial_x^t}{\left(1 - v^2\partial_{xx}^{tt}\right)^{3/2}}\right] P = 0 \,, \tag{24}$$

result. These equations will include transmission coefficient effects, but *do not* add any new ray-tracing or refraction (Snell's law) effects.

Ray-Tracing Effects of the Square-Root Approximation

The one-way wave equations used for wave equation migration are derived from equation (23) by approximating the square-root. This approximation changes the dispersion relation of the one-way equation, and results in incorrect treatment of waves which do not travel in a near-vertical direction. In a variable velocity medium, this is manifest in the incorrect refraction of waves in regions of inhomogeneity or at interfaces. This can be summed up by saying that Snell's law is not obeyed in the migration process.

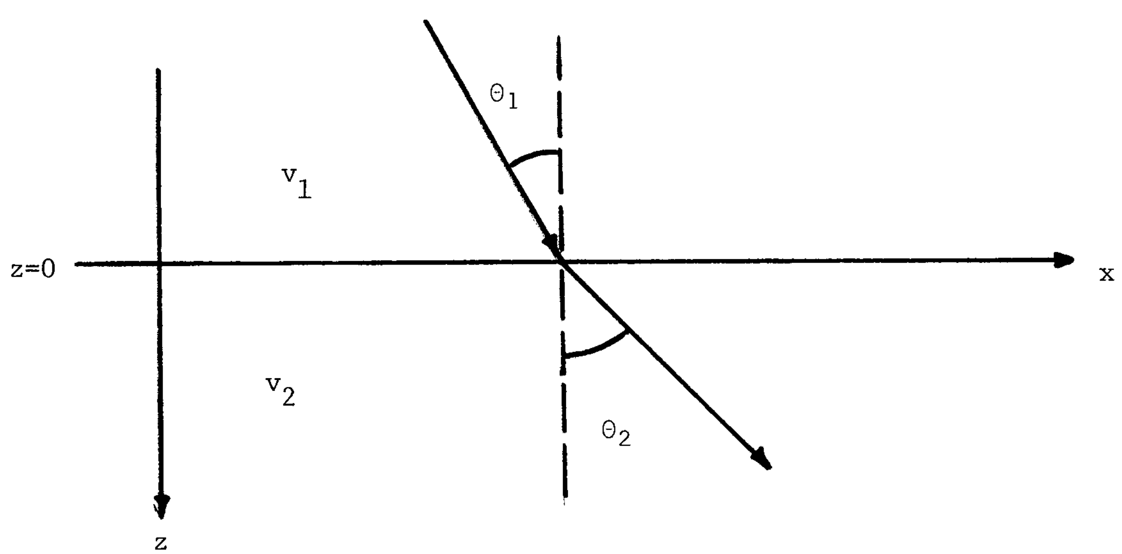
The "15-degree equation" in un-retarded time coordinates can be written as

$$(\partial_{zt} - \frac{1}{v}\partial_{tt} + \frac{v}{2}\partial_{xx})P = 0 \ . \tag{25}$$

(This is derived by taking the first term in the Taylor's series expansion for the square-root in equation (23)).

Consider now the refraction of a wave incident on an interface at z=0. The incident wave will be of the form

$$P_1(x,z,t)=\exp\left[i\omega\left(t - \frac{k_x^{(1)}}{\omega}x - \frac{k_z^{(1)}}{\omega}z\right)\right]$$



and refracted wave of the form

$$P_2(x,z,t) = \exp\left[i\omega\left(t - \frac{k_x^{(2)}}{\omega} x - \frac{k_z^{(2)}}{\omega} z\right)\right].$$

At the interface, $z=0$ and $P_1=P_2$ which implies that $(k_x^{(1)}/\omega) = (k_x^{(2)}/\omega)$. For the true wave equation, or for the one-way wave equation without square root approximation, $k_x = \frac{\omega}{v} \sin\theta$, so this just gives Snell's law:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad . \tag{26}$$

For the 15-degree equation, however vk_x/ω is not $\sin\theta$ because $\omega \neq v\left(k_x^2 + k_z^2\right)^{1/2}$. The dispersion relation for the 15-degree equation can be written

$$\omega^2 - vk_z\omega - \frac{v^2 k_x^2}{2} = 0 \tag{27}$$

from which it follows that

$$\omega = \frac{vk_z}{2}\left[1 + \left(1 + \frac{2k_x^2}{k_z^2}\right)^{1/2}\right] \quad . \tag{28}$$

Then

$$\frac{vk_x}{\omega} = \frac{k_x}{\left(k_x^2 + k_z^2\right)^{1/2}} \quad \frac{v\left(k_x^2 + k_z^2\right)^{1/2}}{\omega} \quad . \tag{29}$$

Combining (28) and (29), and using $k_x/k_z = \tan\theta$,

$$\frac{vk_x}{\omega} = \frac{2\sin\theta}{2\cos\theta\left[1 + \left(1 + 2\tan^2\theta\right)^{1/2}\right]} \tag{30}$$

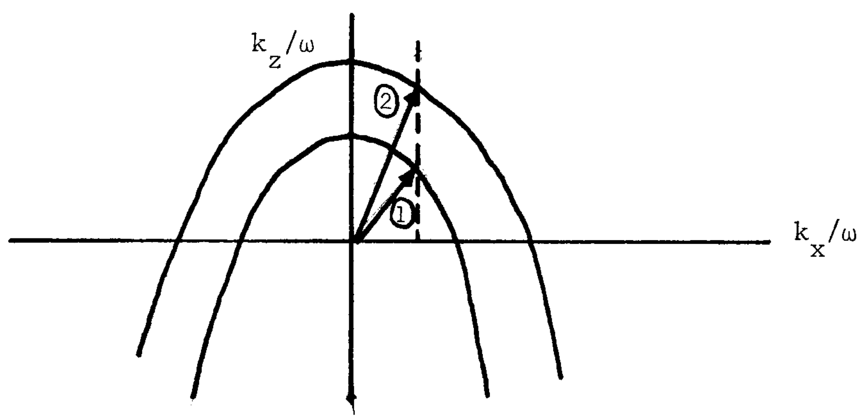
results. Thus for this special case of a horizontal interface, the rule for refraction (Les Hatton's "Cole's law") is

$$\frac{\sin \theta_1}{v_1} \left\{ \cos \theta_2 \left[1 + \left(1 + 2 \tan \theta_2 \right)^{1/2} \right] \right\} =$$

$$\frac{\sin \theta_2}{v_2} \left\{ \cos \theta_1 \left[1 + \left(1 + 2 \tan \theta_1 \right)^{1/2} \right] \right\} \; . \qquad (31)$$

For interfaces at an angle to the horizontal, the refraction rule will be different but can always be determined by remembering that the wave-number component which is parallel to the interface must be conserved.

This whole refraction argument can be remembered quite easily by referring to the sketch below.



It shows two parabolas corresponding to the dispersion relations for the 15-degree equation with velocities v_1 and v_2. When the wave passes from medium 1 to medium 2 through a horizontal interface, (k_x/ω) must remain constant. Thus a wave initially travelling in the direction indicated by the vector marked ① will come out travelling in the direction of ②.

The important thing to understand here is that it is the approximation of the square root in the one-way wave equation which causes this non-Snell's-law refraction, not the dropping of the higher order velocity derivative type terms in the asymptotic expansions of the last section. Rays are traced correctly but with improper amplitude by equation (23); they are traced both incorrectly and with improper amplitude by equation (25).