RECONSTRUCTION OF A WAVEFIELD FROM SLANT STACKS

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The *slant stack* Q of a two dimensional wavefield $P(t,h)$ is defined as

$$Q(t,p) = \int_{-\infty}^{\infty} P(t+ph,h)dh \quad . \tag{1}$$

In particular $P(t,h)$ may represent a common midpoint gather where t is time and h is trace offset from the midpoint; $Q(t,p)$ then represents the summed traces whose zero times have been shifted by an amount $p \cdot h$.

Theoretically P may be regained from Q by the relation

$$P(t,h) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_Q(\omega,p) e^{-i\omega(t-ph)} |\omega| \, d\omega dp \quad . \tag{2}$$

$F_Q(\omega,p)$ is the Fourier transform of $Q(t,p)$ with respect to t .

In the time domain equation (2) may be rewritten as

$$P(t,h) = \int_{-\infty}^{\infty} Q(t-ph,p) \; * \left[\frac{1}{2\pi} M_\Omega \, (t-ph)\right] dp \tag{3}$$

where $*$ is convolution with respect to time, Ω is a cutoff frequency applied to the data, and M_Ω is defined to be

$$M_\Omega(t) \triangleq \Omega \frac{\sin \Omega t}{\pi t} - \frac{2 \sin^2(\Omega t/2)}{\pi t^2} \tag{4}$$

Equations (2),(3),(4) are derived in the following sections.

Derivation

Double Fourier transform $P(t,h)$ into $F_P(\omega,k_h)$:

$$F_P(\omega,k_h) = \iint_{-\infty}^{\infty} P(t,h)e^{i\omega t}e^{ik_h h}\, dt\, dh \tag{5}$$

Change variables of integration:

$$t = t' + ph$$
$$h = h$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1 \quad ,$$

and by the change of variables, equation (5) becomes

$$F_P(\omega,k_h) = \iint_{-\infty}^{\infty} P(t'+ph,h)e^{i\omega(t'+ph)}e^{ik_h h}\,|J|\,dt'dh$$

$$= \iint_{-\infty}^{\infty} P(t'+ph,h)e^{i\omega t'}e^{i(k_h+\omega p)h}\,dt'dh \tag{6}$$

Let $k_h = -\omega p$. Then (6) becomes

$$F_P(\omega,-\omega p) = \iint_{-\infty}^{\infty} P(t'+ph,h)e^{i\omega t'}\,dt'dh$$

$$= \int_{-\infty}^{\infty} Q(t',p)e^{i\omega t'}\,dt'$$

by equation (1). Therefore we have the relation

$$F_P(\omega,-\omega p) = F_Q(\omega,p) \qquad (7)$$

which is similar to the Central Slice Theorem in Fourier Optics Theory. [see Swindell]. $Q(t,p)$ can be thought of as a projection of P onto the time axis $h=0$. Relation (7) then states that the two dimensional Fourier transform of P evaluated along the line $k_h=-\omega p$ is simply the Fourier transform with respect to time of its projection $Q(t,p)$.

To reconstruct P from Q a method similar to rho filtering and back projection in Fourier Optics may be used. Back projection is a method by which an image may be reconstructed from its projections in different directions. The analogous procedure of reconstructing our original wavefield P by "back projecting" the slant stacks is described below.

Slant Stack Inversion

Inverse Fourier transform $F_P(\omega,k_h)$ into $P(t,h)$:

$$P(t,h) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_P(\omega,k_h) e^{-i\omega t} e^{-ik_h h} \, d\omega dk_h \qquad (8)$$

Again change variables of integration [see Apostol sec. 10-9]:

$$\omega = f_1(\omega,p) \triangleq \omega$$

$$k_h = f_2(\omega,p) \triangleq -\omega p \quad ,$$

$$\text{Jacobian } J = \begin{vmatrix} \partial f_1/\partial\omega & \partial f_1/\partial p \\ \partial f_2/\partial\omega & \partial f_2/\partial p \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -p & -\omega \end{vmatrix} = -\omega \; ,$$

so $|J| = |\omega|$.

Then upon substituting the change of variables into (8):

$$P(t,h) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_P(\omega,-\omega p)\, e^{-i\omega t} e^{-i(-\omega p)h} \left|\omega\right| d\omega dp$$

Use (7):

$$P(t,h) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_Q(\omega,p) e^{-i\omega(t-ph)} \left|\omega\right| d\omega dp \qquad (9)$$

which is identical to (2), which was to be proved.

The Jacobian J vanishes over the line $\omega=0$. However since the measure of the subset $\{(\omega,p):\omega=0\}$ is zero, integral (2) remains well defined [Apostol sec. 10-9], even though J vanishes over this set.

The steps taken to reconstruct P from Q using (9) may be summarized:

a) Fourier transform $Q(t,p) \rightarrow F_Q(\omega,p)$

b) Multiply by $|\omega|$ ("Rho Filtering")
Since the inverse Fourier transform of $|\omega|$ doesn't exist, and to ensure convergence of integral (9), we can use in place of $|\omega|$ the filter $|\omega| \sqcap (\omega/2\Omega)$, where

$$\sqcap\left(\frac{\omega}{2\Omega}\right) \triangleq \begin{cases} 1 & |\omega| \leq \Omega \\ 0 & |\omega| > \Omega \end{cases}. \qquad (10)$$

Ω represents a cutoff frequency and may be set to the Nyquist frequency of the sampled data.
Now

$$|\omega| \sqcap \left(\frac{\omega}{2\Omega}\right) = \Omega \sqcap \left(\frac{\omega}{2\Omega}\right) - \Omega \Lambda \left(\frac{\omega}{\Omega}\right) \qquad (11)$$

where

$$\Lambda\left(\frac{\omega}{\Omega}\right) \triangleq \begin{Bmatrix} 1 - |\omega| & |\omega| \leq \Omega \\ 0 & |\omega| > \Omega \end{Bmatrix}.$$

Define

$$M_\Omega(t) = \text{F.T.}^{-1}\left\{|\omega| \sqcap \left(\frac{\omega}{2\Omega}\right)\right\}$$

$$= \frac{\Omega \sin \Omega t}{\pi t} - \frac{2 \sin^2 (\Omega t/2)}{\pi t^2}, \text{ from (11).}$$

c) Inverse Fourier transform $F_Q(\omega,p)[|\omega| \sqcap (\frac{\omega}{2\Omega})]$ to obtain a function $Q'(t-t,p)$.

d) Integrate (or "Back project") to get

$$P(t,h) = \int_{-\infty}^{\infty} \frac{1}{2\pi} Q'(t-ph,p)\, dp \qquad (12)$$

Note (12) is in the form of a reverse slant stack.

With the definition of M_Ω in paragraph (b), see that by the convolution theorem

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_Q(\omega,p) \; |\omega| \; \sqcap \left(\frac{\omega}{2\Omega}\right) e^{-i\omega t}\, d\omega$$

$$= \text{F.T.}^{-1}\left\{F_Q(\omega,p) \cdot |\omega| \; \sqcap \left(\frac{\omega}{2\Omega}\right)\right\}$$

$$= Q(t,p) \; * \; M_\Omega(t) \qquad (*\text{=convolution})$$

With this result (9) may be rewritten as

$$P(t,h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(t-ph,h) \; * \; M_\Omega(t-ph)\, dp$$

which is (3).

REFERENCES

[1] Apostol, T. M., *Mathematical Analysis*, Addison-Wesley, 1957.

[2] Bracewell, Ron, *The Fourier Transform and its Applications*, McGraw-Hill, 1965.

[3] Swindell, William and Harrison H. Barrett, "Computerized Tomography," *Physics Today*, Dec. 1977, pp. 32-41.