MIGRATION IN SLANT-MIDPOINT COORDINATES

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Given the transformation

$$y \quad = \quad (g+s)/2 \tag{1a}$$

$$h \quad = \quad (g-s)/2 \tag{1b}$$

$$t' \quad = \quad t - p(g-s) \tag{2}$$

where (g,s,t) are geophone, shot, and travel time, (y,h,t') are midpoint, half offset, and slanted time, v(z) is velocity as a function of depth, and p is a parameter which turns out to be the Snell's law parameter of the ray about which we will expand. The statement that we have two different mathematical functions to represent the wave field depending on whether we use (t,g,s) or (t',y,h) coordinates is

$$P(t,g,s) \quad = \quad Q(t',y,h) \tag{3}$$

The chain rule for partial differentiation of (3) gives

$$P_t \quad = Q_{t'} \tag{4a}$$

$$P_g \quad = \quad -pQ_{t'} \quad + \quad \frac{1}{2}Q_y \quad + \quad \frac{1}{2}Q_h \tag{4b}$$

$$P_s \quad = \quad pQ_{t'} \quad + \quad \frac{1}{2}Q_y \quad - \quad \frac{1}{2}Q_h \tag{4c}$$

Let Fourier transform variables be defined by the correspondences

$$(t,g,s) \quad \leftrightarrow \quad \exp\ i(-\omega,k_g,k_s) \tag{5a}$$

$$(t',y,h) \quad \leftrightarrow \quad \exp\ i(-\omega',k_y,k_h) \tag{5b}$$

Upon Fourier Transformation the set (4) becomes

$$\omega = \omega' \tag{6a}$$

$$k_g = p\omega' + \frac{1}{2}k_y + \frac{1}{2}k_h \tag{6b}$$

$$k_s = -p\omega' + \frac{1}{2}k_y - \frac{1}{2}k_h \tag{6c}$$

Define some sine like quantities

$$Y = \frac{vk_y}{2\omega} \tag{7a}$$

$$H = \frac{vk_h}{2\omega} \tag{7b}$$

And with (6) we make some more definitions G,S which relate to sines of incidence and departure angles

$$\frac{vk_g}{\omega} = Y + (H + pv) = G + pv \tag{8a}$$

$$\frac{vk_s}{\omega} = Y - (H + pv) = S - pv \tag{8b}$$

It is evident from (8) that choice of non-zero p in equation (2) amounts to an origin shift for incidence and departure angles and for H. We now use the scalar wave equation dispersion relationship to define the vertical wave numbers for downward continuation of geophones k_{z_g} and shots k_{z_s}

$$k_{z_g} = \frac{\omega}{v}\left[1 - \left(\frac{vk_g}{\omega}\right)^2\right]^{\frac{1}{2}} = \frac{\omega}{v}\Phi_g \tag{9a}$$

$$k_{z_s} = \frac{\omega}{v}\left[1 - \left(\frac{vk_s}{\omega}\right)^2\right]^{\frac{1}{2}} = \frac{\omega}{v}\Phi_s \tag{9b}$$

We now assert that the downward continuation of both shots and geophones, i.e. the whole experiment can be done with

$$\frac{dP}{dz} = -ik_z P = -i\frac{\omega}{v}(\Phi_g+\Phi_s)P = \frac{1}{v}\Phi P_t \tag{10}$$

where, using (8) and (9)

$$\Phi = \Phi_g + \Phi_s \tag{10a}$$

$$\Phi = \left[1-\left(G+pv\right)^2\right]^{\frac{1}{2}} + \left[1-\left(S-pv\right)^2\right]^{\frac{1}{2}} \tag{10b}$$

It is now convenient to use (s,c,t) to denote sine, cosine, and tangent. (These will not be used in equations involving shotpoint s, and time t.)

$$\Phi_g = (1-s^2-2sG-G^2)^{1/2} \tag{11a}$$

$$= c(1-2tG-G^2/c^2)^{1/2} \tag{11b}$$

Let

$$R = (1-2tG-G^2/c^2)^{1/2} \tag{12}$$

Now recall the Muir-Engquist rational expansion of square root R

$$R_{j+2} = 1 + \frac{-2tG-G^2/c^2}{1 + R_j} \tag{13}$$

First consider even orders

$$R_o = 1 \tag{14a}$$

$$R_2 = 1 -tG - \frac{1}{2c^2} G^2 \tag{14b}$$

More interesting are the odd orders. Rearrange (12)

$$R = [(1-tG)^2 - (c^{-2} + t^2) G^2] \tag{15}$$

Inspection of (11b) for very small G^2 (but not necessarily very small tG) suggests starting (13) with

$$R_1 = 1-tG \tag{16}$$

Rearrange (13)

$$R_{j+2} = 1-tG + \frac{tG(-1+R_j) - G^2/c^2}{1 + R_j} \tag{17}$$

To avoid algebraic clutter we develop R in the form

$$R_j = 1-tG + \frac{N_j}{D_j} \tag{18}$$

Inserting into (17) we get

$$\frac{N_{j+2}}{D_{j+2}} = \frac{tG(-tG + N_j/D_j) - G^2/c^2}{2-tG + N_j/D_j}$$

$$= \frac{tGN_j - (c^{-2}+t^2)G^2D_j}{D_j(2-tG) + N_j} \tag{19}$$

Starting with $N_1=0$, $D_1=1$ the numerator and denominator recurrences in (19) give

$$R_1 = 1 - tG \tag{20a}$$

$$R_3 = 1 - tG - \frac{(c^{-2}+t^2)G^2}{(2 - tG)} \tag{20b}$$

$$R_5 = 1 - tG - \frac{2(c^{-2}+t^2)G^2}{4-4tG-G^2/c^2} \tag{20c}$$

Now that we have completed the algebraic work we can perform substitutions. Consider for example the third order approximant of Φ , let $c^{-2}+t^2 = a$

$$\Phi^{(3)} = \Phi_g^{(3)} + \Phi_s^{(3)}$$

$$= 2-2tH - \frac{a(Y+H)^2}{2-t(Y+H)} - \frac{a(Y-H)^2}{2+t(Y-H)} \qquad (21)$$

This expression becomes fourth order in H if we try to rationalize the denominator. Thus if we intend to stick with tridiagonal schemes and non-zero H then the highest scheme we can use is given by R_2

$$\Phi^{(2)} = 2-2tH - \frac{1}{2c^2}\ [(Y+H)^2 + (Y-H)^2] \qquad (22)$$

which amounts to the differential equation, from (7), (10), (22)

$$\frac{dP}{dz} = \frac{\cos}{v}\ \partial_{t'} \left[2 - \frac{s}{c}\frac{v}{2}\partial_h^{t'} - \frac{v^2}{4c^2}\left(\partial_{yy}^{t't'} + \partial_{hh}^{t't'}\right)\right] P \qquad (23)$$

The case of particular interest is H=0. The zero spatial frequency in h in the space of (h,t') clearly represents a line integral across the (h,t) data in the h direction at each value of t' = t-p2h. These are slant stacks of common midpoint gathers. Specializing (21) to H=0 we get

$$\Phi^{(3)} = 2 - (c^{-2}+t^2)Y^2 \left(\frac{1}{2-tY} + \frac{1}{2+tY}\right)$$

$$= 2 - \frac{(c^{-2}+t^2)\ Y^2}{1 - \frac{1}{4}t^2Y^2} \qquad (24)$$

Neglecting the time shift term 2 this third order downward continuation is

$$\left(1 - \frac{v^2s^2}{16c^2}\partial_{yy}^{t't'}\right) P'_{zt'} = -\frac{v}{4}\ \frac{1+s^2}{c}\ P'_{yy} \qquad (25)$$

It seems that the fifth order scheme (20c) could also be used in a purely tridiagonal framework if the z-steps were alternated between G and S. In a constant velocity media $t'=t_o\cos\theta$ where t_o is vertical incidence time. Then at small dips

$$P'_{zt_o} = -\frac{v}{4}\ (1+s^2)\ P'_{yy} \qquad (26)$$

which is like FGDP p.253 equation (11-3-18) except that sine in (26) is like tangent in FGDP. Also in FGDP the tangent squared term, in present notation is written as $2h/vt_o$ and the equation is interpreted as a migration before stack equation. While the methods of the present paper allow for the development of very accurate migration schemes for slant midpoint stacks in velocity stratified media. The FGDP development was done with constant velocity media and hyperbolic moveout. Thus the neglect of $H \leftrightarrow \partial_h$ seems more realistic in FGDP provided that the dips are modest.