VARIABLE VELOCITY: WAVE EXTRAPOLATION AND REFLECTION

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Let us consider the scalar wave equation

$$P_{tt} = v(x,z)^2(P_{xx} + P_{zz}); \quad t,z \geq 0, \tag{1}$$

where the velocity v depends on both x and z.

We will analyze two related problems. First, we assume smooth velocity and study the effect of velocity variation on approximations of (1), that are used for extrapolating the solution in space. We will see how a paraxial approximation of the wave equation can be modified for nonconstant v. The main changes, locally, will be in the amplitude of the waves.

The second problem regards coupling of downgoing and upcoming waves at a dipping interface. Both waves are assumed to be described by one-way wave equations. This analysis can be applied to the study of multiples and multiple suppression.

Let us now assume that (1) is given with a smooth $v(x,z)$. We will work with positive time and assume that P is given at $z=0$ and that P and P_t vanish for $t=0$.

We start from the constant coefficient 15° paraxial approximation of (1) describing downgoing waves:

$$[\partial_z + (\partial_t/v) - (v/2)\partial^t_{xx}]P = 0. \tag{2}$$

The results can also be used for retarded time as in the migration equation. Upper index, like ∂^t, is here used to denote integration; lower index denotes derivatives:

$$\partial^t P = \int_{-\infty}^{t} P\, dt.$$

We propose the modified equation

$$[\partial_z + (\partial_t/v) - (v/2)\partial^t_{xx} + Q]P = 0, \quad (3)$$

where Q is an operator to be determined such that a solution P to (3) is as close as possible a solution to (1) when v is variable. We mean here by "close" that the error shall be small when the derivatives of v and the horizontal derivatives of P are small.

When (1) is written in the form

$$\left[(\partial_{tt}/v^2) - \partial_{xx} - \partial_z\partial_z\ P = 0\right] \quad (4)$$

we can substitute $\partial_z P$ using (3). Note that we cannot substitute the operator ∂_z,

$$\partial_z = -(\partial_t/v) + (v/2)\partial^t_{xx} - Q, \quad (5)$$

directly for both the ∂_z in (4) since ∂_z does not commute with v. The substitution gives

$$\{(\partial_{tt}/v^2) - \partial_{xx} - \partial_z[-(\partial_t/v) + (v/2)\partial^t_{xx} - Q]\}\ P = 0,$$

$$[(\ _{tt}/v^2) - \partial_{xx} - (v_z/v^2)\partial_t + (\partial_t/v)\partial_z - (v_z/2)\partial^t_{xx} - (v/2)\partial^t_{xx}\partial_z$$

$$+ \partial_z Q]P = 0.$$

Here we can substitute for $\partial_z P$ again, and after a few algebraic steps, we get

$$\left(-\frac{v_z}{v^2}\partial_t - \frac{v_{xx}}{2v} + \frac{v_x^2}{v^2} - \frac{v_x}{v}\partial_x - \frac{v_z}{2}\partial^t_{xx} - \frac{vv_{xx}}{4}\partial^{tt}_{xx} - \frac{vv_x}{2}\partial^{tt}_{xxx}\right.$$

$$\left. - \frac{v^2}{4}\partial^{tt}_{xxxx} - \frac{1}{v}\partial_t\ Q + \frac{v}{2}\partial^t_{xx}Q + \partial_z Q\right) P = 0. \quad (6)$$

Let us set the level of ambition such that we neglect terms with more derivatives than two in the x-direction and more integrations than one in the t-direction. If we transform each factor in each term separately, this means that we drop terms of higher order than k_x^2 or $1/\omega$ (k_x, k_z, and $-\omega$ are the dual of x, z, and t). We further restrict Q to be at most first-order in x and to have the form

$$Q \;=\; q_1 + q_2 \partial_x^t + q_3 \partial^t, \tag{7}$$

where q_i are functions of x and z. The equation (6) will then be

$$-\frac{v_z}{v^2}\partial_t - \frac{v_{xx}}{2v} + \frac{v_x^2}{v^2} - \frac{v_x}{v}\partial_x - \frac{v_z}{v}\partial_{xx}^t - \frac{2q_1}{v}\partial_t - \frac{2q_2}{v}\partial_x - \frac{2q_3}{v} + \frac{q_2 v_x}{v^2}$$

$$+ \frac{v}{2}\,[2q_1\partial_{xx}^t + 2(q_1)_x\partial_x^t + (q_1)_{xx}\partial^t] + (q_1)_z + (q_2)_z\partial_x^t + (q_3)_z\partial^t$$

$$- q_1(q_1 + q_2\partial_x^t + q_3\partial^t) - q_2(q_1)_x\partial^t - q_2 q_1\partial_x^t - q_3 q_1\partial^t \;=\; 0. \tag{8}$$

We group the terms with the same differential operator together:

$$-\frac{v_z}{v^2} - \frac{2q_1}{v} \quad \partial_t \;=\; 0,$$

$$-\frac{v_x}{v} - \frac{2q_2}{v} \quad \partial_x \;=\; 0,$$

$$-\frac{v_{xx}}{2v} + \frac{v_x^2}{v^2} - \frac{2q_3}{v} + (q_1)_z - (q_1)^2 + \frac{q_2 v_x}{v^2} \;=\; 0.$$

These equations determine the q_i:

$$q_1 \;=\; -\,v_z/2v\,,$$

$$q_2 \;=\; -\,v_x/q\,, \tag{9}$$

$$q_3 \;=\; -\frac{v_{xx}}{4} + \frac{v_x^2}{4v} - \frac{V_{zz}}{4} + \frac{V_z^2}{8v}$$

With our choice of Q we cannot delete the factors in front of the other differential operators. The effect of these non-zero terms is, however, of higher order. If the derived value of Q is inserted into (3), we have

$$P_{tz} + (1/v)P_{tt} - (v/2)P_{xx} - (v_z/2v)P_t - \frac{1}{2}v_x P_x$$

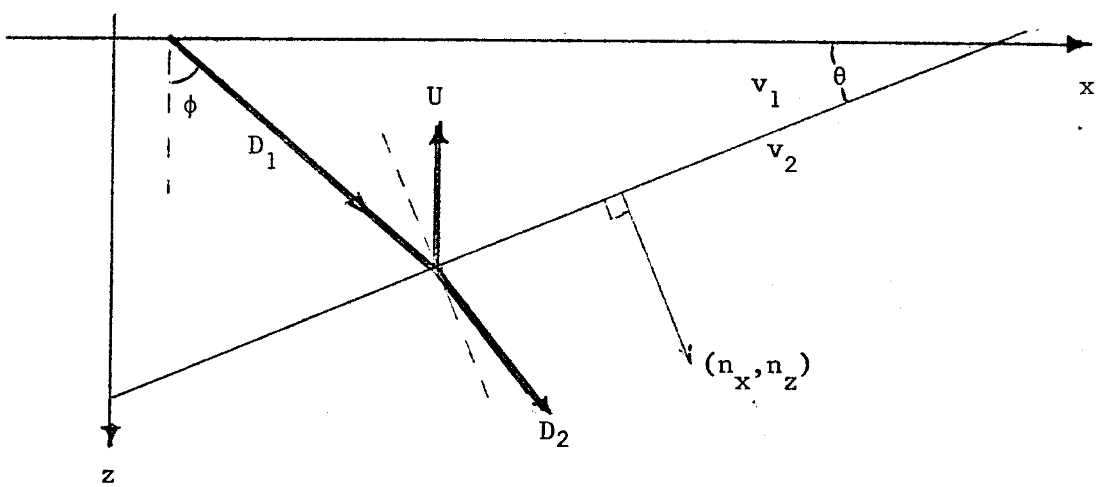
$$+ [-(v_{xx}/4) - (v_x^2/4v) - (v_{zz}/4) + (v_z^2/8v)]P = 0, \qquad (10)$$

The way in which higher-order terms are neglected can be chosen depending on the size of the functions P and v and their derivatives in a particular situation. In a simplified version of equation (10),

$$P_{tz} + (1/v)P_{tt} - (v/2)P_{xx} - \frac{1}{2}(v_z/v)P_t - \frac{1}{2}v_x P_x = 0,$$

we can identify the new terms. The amplitude of the transmitted wave is controlled by $(v_z/v)P_t$ in the z-direction and by $v_x P_x$ in the x-direction.

Let us now turn to the problem of coupling downgoing and upcoming waves at a dipping interface. We assume that the velocity has one constant value (v_1) above the interface and another (v_2) below. The discussion below is in many respects the same as that in Claerbout's book.



The incident downgoing wave is denoted D_1, the transmitted wave D_2, and the reflected wave U. The normal to the interface is $(n_x, n_z) =)\sin\theta, \cos\theta)$. We introduce new coordinates (n,s) where n is normal to the interface and s is tangent:

$$n = n_x x + n_z z,$$
$$s = n_z x - n_x z, \tag{11}$$
$$x = n_x n + n_z s,$$
$$z = n_z n - n_x s.$$

The wave equation in the new coordinates is then

$$(1/v^2)P_{tt} = P_{nn} + P_{ss}, \tag{12}$$

with the interface conditions

$$P, P_n \quad \text{continuous.} \tag{13}$$

With the following form of the waves,

$$D_1 = \exp[i(-\omega t + \alpha_1 n + \beta s)],$$
$$D_2 = T\exp[i(-\omega t + \alpha_2 n + \beta s)],$$
$$U = R\exp[i(-\omega t - \alpha_1 n + \beta s)],$$

we can determine T and R from the differential equation (12) and the interface conditions (13):

$$\omega^2 = v_1^2(\alpha_1^2 + \beta^2),$$
$$\omega^2 = v_2^2(\alpha_2^2 + \beta^2),$$
$$1 + R = T,$$
$$\alpha_1 - R\alpha_1 = T\alpha_2 .$$

Let us relate the wave numbers α_1, α_2 and β with the wave numbers in the original coordinate system:

$$\alpha_1 n_x + \beta n_z = k_x,$$
$$\alpha_1 n_z - \beta n_x = k_z,$$

$$\alpha_1 = n_x k_x + n_z k_z$$

$$= \omega\{n_x(k_x/\omega) + (n_z/v_1)[1 - (v_1^2 k_x^2/\omega^2)]^{1/2}\},$$

$$\beta = n_z k_x - n_x k_z$$

$$= \omega\{n_z(k_x/\omega) - (n_x/v_1)[1 - (v_1^2 k_x^2/\omega^2)]^{1/2}\} .$$

These formulae lead to expressions of the reflection and transmission coefficients R and T in terms of v_1, v_2, θ and k_x/ω. We prefer to describe the direction of the downgoing wave with k_x and ω rather than with the angle ϕ. In this way it is easier to see what the reflection will be for a general downgoing wave field:

$$R = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1 - [(\omega^2/v_2^2) - \beta^2]^{1/2}}{\alpha_1 + [(\omega^2/v_2^2) - \beta^2]^{1/2}} = \frac{A - B}{A + B}, \tag{14}$$

where

$$A = \sin\theta(k_x/\omega) + (\cos\theta/v_1)[1 - (v_1^2 k_x^2/\omega^2)]^{1/2} ,$$

$$B = \left\{\frac{1}{v_2^2} - \left[\cos\theta\frac{k_x}{\omega} - \frac{\sin\theta}{v_1}\left(1 - \frac{v_1^2 k_x^2}{\omega^2}\right)^{1/2}\right]^2\right\}^{1/2}$$

The reflection coefficient R can hence be written as a series in powers of k_x/ω:

$$R(v_1,v_2,\theta,k_x/\omega) = r_0(v_1,v_2,\theta) + r_1(v_1,v_2,\theta)(k_x/\omega) + r_2(v_1,v_2,\theta)(k_x^2/\omega^2) + \ldots \tag{15}$$

If we define a through

$$a = [(v_1/v_2)^2 - \sin^2\theta]^{1/2},$$

the first terms in the expansion (15) will be

$$r_0 = (\cos\theta - a)/(\cos\theta + a),$$

$$r_1 = \frac{2v_1 \sin\theta}{a} \frac{(a - \cos\theta)}{(a + \cos\theta)}$$

When we use ϕ to describe the direction of D_1, we have

$$v_1 k_x/\omega = \sin\theta$$

$$A = \frac{\sin\theta \sin\theta}{v_1} + \frac{\cos\theta}{v_1}(1 - \sin^2\phi)^{1/2}$$

$$= \cos(\phi - \theta)/v_1,$$

$$B = \left\{ \frac{1}{v_2^2} - \left[\frac{\cos\theta\sin\theta}{v_1} - \frac{\sin\theta}{v_1}(1 - \sin^2\phi)^{1/2} \right]^2 \right\}^{1/2}$$

$$= [(1/v_2^2) - (\sin^2(\phi - \theta)/v_1^2)]^{1/2}$$

That is,

$$R(v_1, v_2, \theta, \sin\theta/v_1) = r_0(v_1, v_2, \theta - \phi).$$

Let us now use formula (15) and go back to the physical domain. The number k_x/ω is replaced by the operator $-\partial_x^t$. This means that the upcoming wave field U is given the following initial data at the interface:

$$U = (r_0 - r_1 \partial_x^t + r_2 \partial_{xx}^{tt} + \ldots) D_1.$$

Let the downgoing and upcoming waves be described by one-way wave equations:

$$(\partial/\partial z)D = L_+ D,$$

$$(\partial/\partial z)U = L_- U.$$

For the 15° approximation, we have

$$L_\pm = \mp (1/v)\partial_t \pm (v/2)\partial_{xx}^t.$$

The coupling can then be written

$$\partial U/\partial z \quad = \quad L_{-}U + \delta_I(r_0 - r_1 \partial_x^t + \ldots)D,$$

where δ_I is a delta-function along the interface. With the 15° approximation and two terms in the expansion, we have

$$U_{tz} - (1/v)U_{tt} + (v/2)U_{xx}$$

$$= \delta_I(r_0 D_t - r_1 D_x).$$

If retarded time is used for both U and D, the equation becomes

$$U_{tz} + \frac{v}{2} U_{xx} = \delta_I \left[r_0 D_t(t - \frac{2z}{v_1}, x, z) - r_1 D_x(t - \frac{2z}{v_1}, x, z) \right] .$$

Other approximations of the one-way wave equation or the use of other coordinate frames do not change the structure of the result.