DISPERSION RELATIONS FOR ELASTIC WAVES

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In SEP-10, two paraxial approximations of the elastic wave equation were introduced. The first of these (SEP-10, pp. 125-140), is order $|k_x/\omega|^3$ and is cast in terms of horizontal and vertical displacement fields

$$\underline{u}_{tz} + \begin{pmatrix} 1/\beta & 0 \\ 0 & 1/\alpha \end{pmatrix} \underline{u}_{tt} + (\beta - \alpha) \begin{pmatrix} 0 & 1/\beta \\ 1/\alpha & 0 \end{pmatrix} \underline{u}_{tx} + \frac{1}{2} \begin{pmatrix} \beta-2\alpha & 0 \\ 0 & \alpha-2\beta \end{pmatrix} \underline{u}_{xx} = 0 , \qquad (1)$$

where

$$\underline{u} = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \text{horizontal disp. field} \\ \text{vertical disp. field} \end{pmatrix},$$

and α and β are the compressional and shear velocities, respectively. The second paraxial approximation (SEP-10, pp. 165-170) is order $|k_x/\omega|^4$ but uses a rather peculiar set of state variables.

$$\underline{r}_{tz} + \begin{pmatrix} \frac{1}{\alpha} & 0 \\ \frac{-2\rho(\alpha-\beta)\beta}{\alpha} & \frac{1}{\beta} \end{pmatrix} \underline{r}_{tt} + \begin{pmatrix} \frac{(\alpha-2\beta)^2}{2\alpha} & \frac{1}{\rho}\frac{\alpha-\beta}{\alpha\beta} \\ \frac{-\rho\beta^2(4\beta-\alpha)(\alpha-\beta)}{\alpha} & \frac{\beta}{2\alpha}(5\alpha-4\beta) \end{pmatrix} \underline{r}_{xx} = 0, \qquad (2)$$

where

$$\underline{r} = \begin{pmatrix} u \\ \frac{\partial}{\partial x}\tau_{zz} \end{pmatrix} = \begin{pmatrix} \text{horizontal disp. field} \\ \partial/\partial x \text{ of normal stress} \end{pmatrix}.$$

To obtain Eq. (2), we algebraically solved the equations given on pp. 169 and 170 of SEP-10 for the elements of the $L^{1/2}$ and C matrices. To make sense of those equations it is necessary to correct two "typos." In the expression for a_{21} at the top of p. 169, the "λ" should be a "γ", and the "$b^{1/2}$" should be a "b" in the expression for ℓ_2 at the bottom of that page.

In this paper, we compare the dispersion relations for the two approximations. The dispersion relations in this case indicate how well P and S waves are modelled along a ray path in the positive z-direction. We have also included in the comparison the dispersion relations of the absorbing boundary conditions proposed by Lysmer and Kuhlemeyer [1969, "A finite dynamic model for infinite media," *J. Eng. Mech. Div. ASCE*, EM4, pp. 859-877]. This boundary condition consists of the viscous damping of shear and normal stress components

$$\tau_{xz} = -\rho\,\beta\,u_t,$$

$$\tau_{zz} = -\rho\,\alpha\,w_t.$$

These equations may be converted to a paraxial approximation by expressing the stress in terms of displacements.

$$\underline{u}_z + \begin{pmatrix} \frac{1}{\beta} & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \underline{u}_t + \begin{pmatrix} 0 & 1 \\ \frac{\alpha^2 - 2\beta^2}{\alpha^2} & 0 \end{pmatrix} \underline{u}_x = 0. \qquad (3)$$

The dispersion relations are found by first Fourier transforming the equations in all components, and then converting to the form

$$L\,\tilde{\underline{u}} = 0.$$

The equations will have non-trivial solutions if and only if $\det L = 0$, and the values of k_z/ω and k_x/ω which make this true define the dispersion relations.

After a little algebra, the following dispersion relations were found:

for Eq. (1):

$$\left(Z - \frac{1}{\alpha} + \frac{\alpha}{2} X^2\right)\left(Z - \frac{1}{\beta} + \frac{\beta}{2} X^2\right) - \frac{1}{2}(\alpha - \beta)^2 X^4 = 0; \qquad (4)$$

for Eq. (2):

$$\left(Z - \frac{1}{\alpha} + \frac{\alpha}{2} X^2\right)\left(Z - \frac{1}{\beta} + \frac{\beta}{2} X^2\right) = 0; \qquad (5)$$

and for Eq. (3):

$$\frac{\left[Z - \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\alpha}\right)\right]^2}{\left[\frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\right]^2} - \frac{X^2}{\frac{(\alpha - \beta)^2}{4\beta^2(\alpha^2 - 2\beta^2)}} = 1, \qquad (6)$$

where for convenience we have let $Z = k_z/\omega$ and $X = k_x/\omega$.

Equation (5) defines a pair of parabolas each of which is equivalent to the 15° scalar approximation for velocities α and β. The basic shape of these curves is independent of the velocity ratio α/β.

Equation (6) is a hyperbola which, along with the dispersion relations for Eq. (5), is shown in Fig. 1. It is obvious from this figure that there are problems with modelling shear waves with Eq. (3).

The dispersion relations implied by Eq. (4) depend on the velocity ratio α/β. For values of this ratio close to unity, the dispersion relations resemble those of Eq. (5) (i.e., a pair of parabolas). However, for larger values of α/β the dispersion relation for shear waves becomes a poorer approximation to the semicircle. In Fig. 2 the dispersion relations are shown for $\alpha/\beta = \sqrt{3}$, and in Fig. 3 for $\alpha/\beta = 3$. For reference in Figs. 2 and 3, the parabolas of Eq. (5) are also included.

The dispersion relations shown in Figs. 1 through 3 indicate that Eq. (1) gives the best representation for P waves and Eq. (2) is best for S waves. Since the shape of the dispersion curves for Eq. (2) is independent of the velocity ratio α/β, it gives the best uniform approximation for both P and S waves.

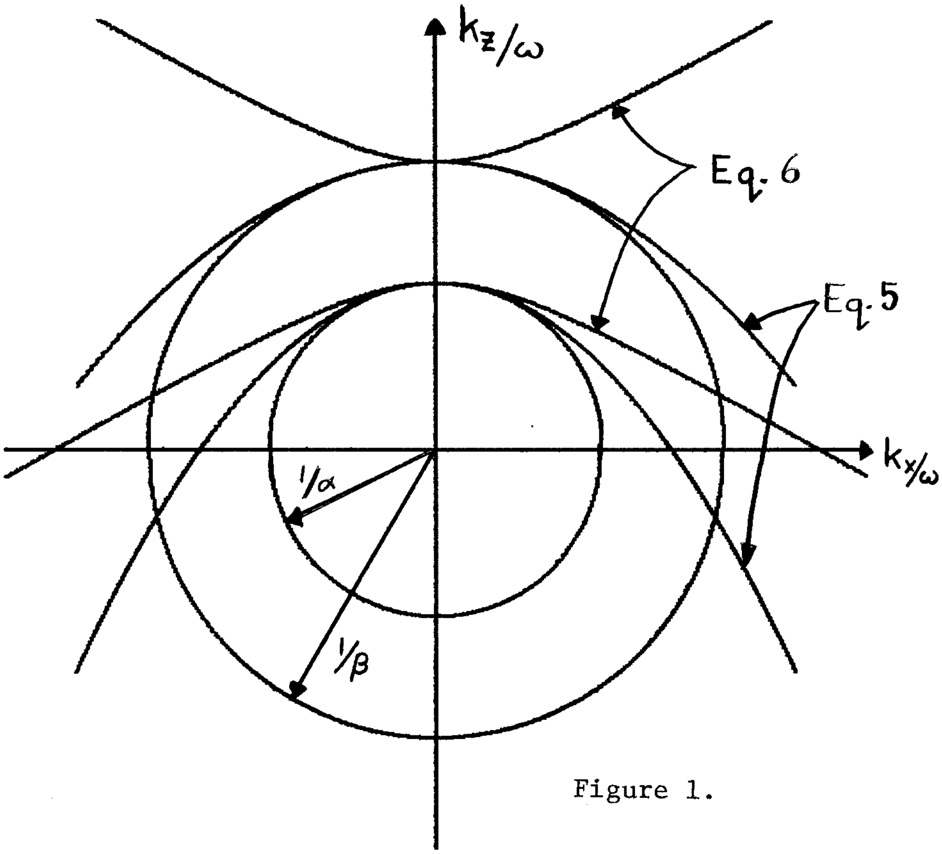

Figure 1.

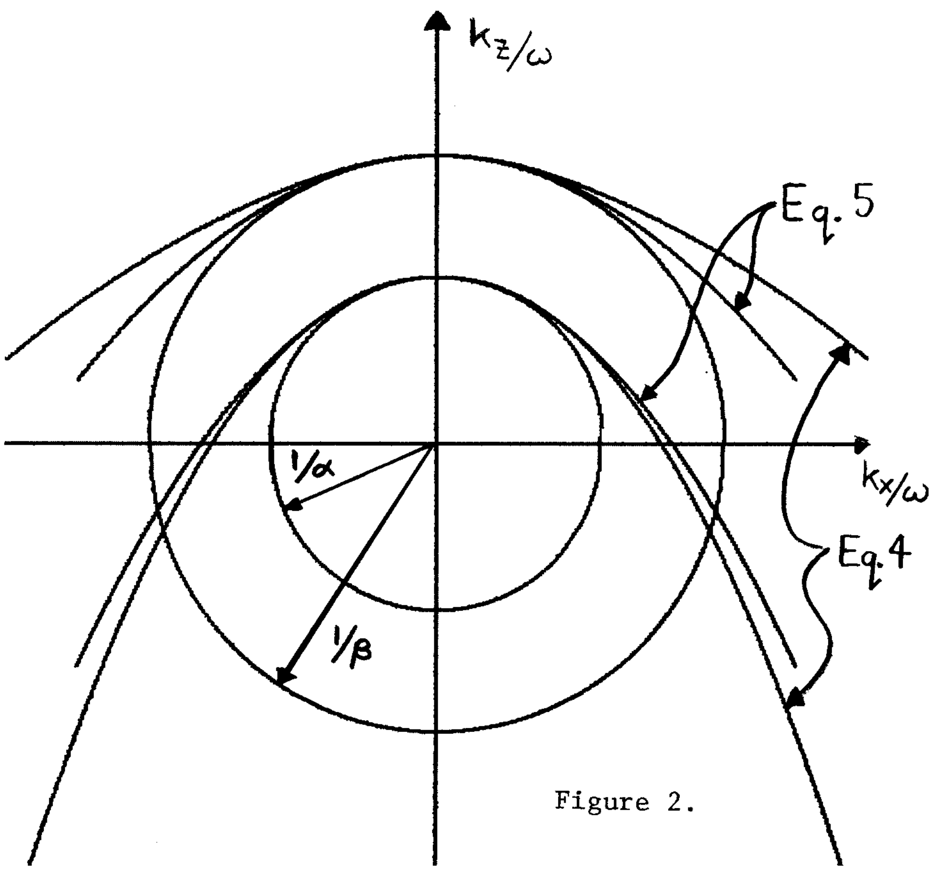

Figure 2.

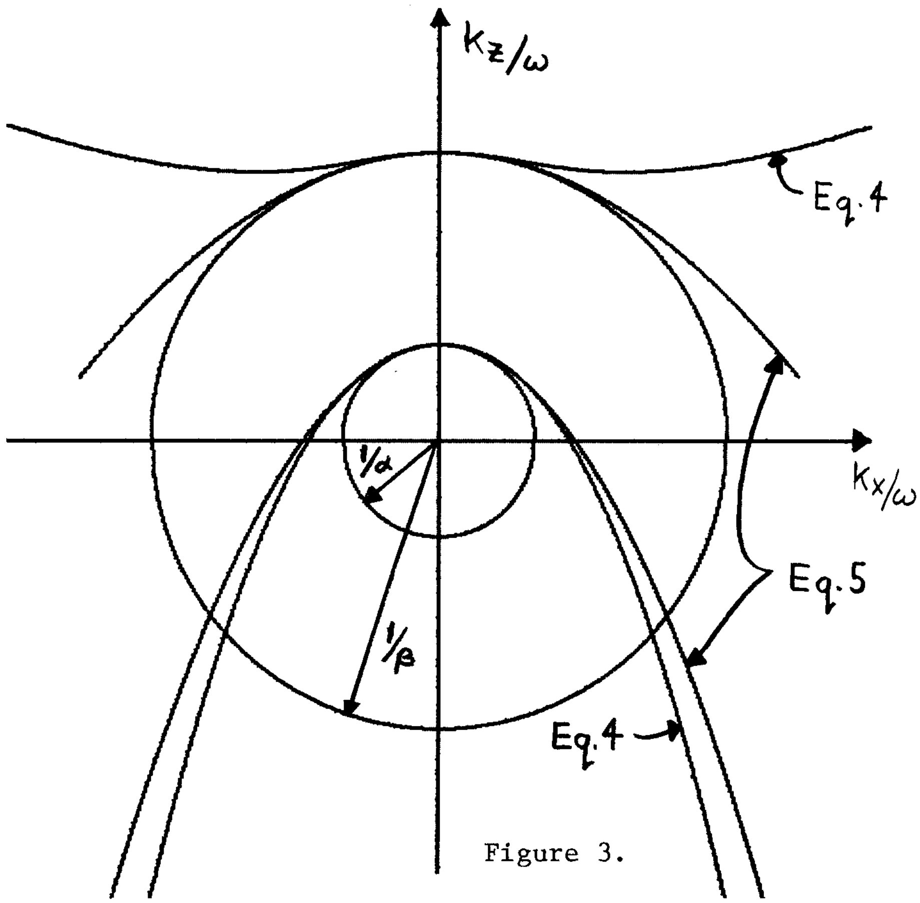

Figure 3.