THE STABILITY OF ABSORBING SIDE BOUNDARIES

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We have received a few telephone calls from people who had stability problems with absorptive side boundary conditions described in SEP-10. First, we will have a problem reformulation that may eliminate the problem. Then we will have a proof that the differential equations with the differential boundary conditions are energy dissipative. This means that any remaining problems must arise from discretization of the differential equations. Then we will incorporate discretization of the x-axis and determine the correct form for energy-dissipative side boundaries. Finally, we'll admit that we really wanted to discretize not only x but also z and t and then prove stability, but couldn't do it within the time allotted to the problem.

First we take up the reformulation which might eliminate boundary instability as a practical problem. The internal equation is

$$P_{zt} = a\,P_{xx}. \tag{1}$$

Simple absorbing boundaries in SEP-10 took the form

$$P_z = a\,b\,P_{xt}. \tag{2}$$

Differentiate (2) with respect to t :

$$P_{zt} = a\,b\,P_{xt}. \tag{3}$$

Substitute P_{zt} from (1) into (3). Integrate with respect to x and divide out a . We get

$$P_x = b\,P_t. \tag{4}$$

Equation (4) is an absorbing boundary condition equivalent in principle to (2). But there may be a practical distinction because (2) involved a z derivative but (4) does not. In other words, use of (2) may require a correct initialization at $z = 0$. With (4) the question of initialization never arises.

Now let us go on to the proof that the internal equation (1) with the side boundary conditions (4) is dissipative (provided b takes the correct sign). Integrating (1) from $-\infty$ to time t, we write

$$P_z = a P^t_{xx} . \tag{5}$$

To look at the amount of energy absorbed by the boundaries (and possibly in the interior), we can consider the energy decay rate with depth, which we will define as

$$Q = -\frac{1}{a}\frac{d}{dz}\int_{-\infty}^{+\infty} dt \int_{x_0}^{x_N} dx\, \frac{1}{2} P^2 \tag{6}$$

$$= -\frac{1}{a}\int dt \int dx\, P\, P_z . \tag{7}$$

Substituting from (5),

$$Q = -\int dt \int dx\, P\, P^t_{xx}$$

Recall the method of integration by parts:

$$uv\Big|_{x_0}^{x_N} = \int_{x_0}^{x_N} v\, du + \int_{x_0}^{x_N} u\, dv . \tag{9}$$

Let $u = P$ and $dv = P^t_{xx}\, dx$. Then (8) becomes

$$Q = -\int dt \left[P\, P^t_x\right]_{x_0}^{x_N} + \int dt \int_{x_0}^{x_N} dx\, P_x\, P^t_x . \tag{10}$$

But the right-hand-side integral, which represents internal absorption, can be written in the form

$$\int dx \int_{-\infty}^{+\infty} \frac{1}{2}\frac{d}{dt}(P^t_x)^2\, dt .$$

Now if we take the wave field P (more precisely, the time integral of the horizontal space derivative P_x^t) to vanish at plus and minus infinite time, then the internal absorption vanishes, leaving us with the boundary absorption

$$Q = -\int_{-\infty}^{+\infty} \left[P \, P_x^t\right]_{x_0}^{x_N} dt \,. \tag{11}$$

Now integrate the boundary condition (4) with respect to time,

$$P_x^t = b\,P, \tag{12}$$

define $b(x_0) = b_0$, $b(x_N) = b_N$, and insert into (11):

$$Q = \int_{-\infty}^{+\infty} [b_0 P(x_0,t)^2 - b_N P(x_N,t)^2] \, dt \,. \tag{13}$$

This is a positive definite quadratic form if and only if $b_0 > 0$ and $b_N < 0$. (This is for diffraction. Migration will imply the opposite signs.) This proves that, as a matter of principle, there is no boundary stability problem.

Now let us discretize the problem with respect to the horizontal x-axis. Rather than start off asserting that we know what an absorbing boundary condition is [as we did with (2)], let us derive the expression for dissipation and see what boundary conditions will keep it positive. First we will need to review the equation for summation by parts. We start with the algebraic identity

$$u_{k+1}v_{k+1} - u_k v_k = v_{k+1}(u_{k+1} - u_k) + u_k(v_{k+1} - v_k) \,. \tag{14}$$

Sum from $k=1$ to $k=N$,

$$u_{N+1}v_{N+1} - u_1 v_1 = \sum_{k=1}^{N} v_{k+1}(u_{k+1} - u_k) + \sum_{k=1}^{N} u_k(v_{k+1} - v_k) \,. \tag{15}$$

Take Eq. (5), discretize the x-axis, $x_k = k\,\Delta x$, and use k as a subscript to denote position on the x-axis. We get

$$P_z = a(P_{k+1}^t - 2P_k^t + P_{k-1}^t) \,. \tag{16}$$

Now define the energy dissipation Q like (6), but with summation over x

instead of integration,

$$Q = -\frac{1}{a}\frac{d}{dz}\int_{-\infty}^{+\infty} dt \sum_{k=1}^{N} \frac{1}{2} P_k{}^2 \tag{17}$$

$$= -\frac{1}{a} \int dt \sum_k P\,P_z . \tag{18}$$

Substituting (16) into (18) gives

$$Q = -\int dt \sum_k P_k(P^t_{k+1} - 2\,P^t_k + P^t_{k-1}) \ . \tag{19}$$

Next we make the definitions

$$u_k = P_k \ , \tag{20}$$

$$v_k = P^t_k - P^t_{k-1} \ , \tag{21}$$

and substitute into (19),

$$Q = -\int dt \sum_k u_k(v_{k+1} - v_k). \tag{22}$$

Apply Eq. (15),

$$Q = \int dt \left[u_1 v_1 - u_{N+1} v_{N+1} + \sum_{k=1}^{N} v_{k+1}(u_{k+1} - u_k)\right] \ . \tag{23}$$

Substitute (20) and (21) into (23),

$$Q = \int dt \left[P_1(P^t_1 - P^t_0) - P_{N+1}(P^t_{N+1} - P^t_N) + \sum_{k=1}^{N} dt \frac{1}{2}\frac{d}{dt}(P^t_{k+1} - P^t_k)^2\right]. \tag{24}$$

Assuming as before that P^t_x vanishes at plus and minus infinite time, we see the right-hand integral vanishes, leaving us with

$$Q = \int dt[P_1(P^t_1 - P^t_0) - P_{N+1}(P^t_{N+1} - P^t_N)] \ . \tag{25}$$

Obviously a dissipative boundary condition at $x = x_0$ is given by

$$P_1 = b(P_1^t - P_0^t) , \tag{26}$$

for all positive values of b .

As stated earlier, we have been unable to determine a precise statement of the difference form of dissipative side boundary conditions where all of x, z, and t are discretized. But at the same time, we believe there is no practical problem with stability. We have noted in more careful examination of the scale factors involved on the cover of SEP-10 that energy actually does flow in at the right-side boundary. But it flows away faster than it flows in. This, of course, raises the interesting question of determining side boundary conditions that allow energy to flow inward in a deliberate, controlled fashion.