INHOMOGENEOUS BOUNDARY CONDITIONS FOR WAVE EQUATIONS

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When solving the wave equation numerically, one usually poses some homogeneous boundary condition, i.e., some function of the wave function and its derivatives is set equal to zero along the calculation boundaries in x. The most commonly used examples of such boundary conditions are the Neumann (zero-slope) and Dirichlet (zero-value) conditions, which correspond physically to totally reflecting boundaries. Alternatively, one might specify some partially absorbing boundary condition, such as one of those suggested by Engquist and Clayton (SEP-10). In this paper, the possibility of posing inhomogeneous boundary conditions will be investigated. Such boundary conditions might correspond physically to source waveforms along the "side" boundary as opposed to source waveforms along the "top" boundary that are introduced via the initial conditions. Several boundary conditions of this sort will be suggested, and lastly, some qualitative comments on the stability of boundary conditions with respect to roundoff error will be made.

We shall consider here solutions of the wave equation

$$P_{xx} + P_{zz} = (1/v^2)P_{tt} \,, \tag{1}$$

in the region

$$t \geq 0, \quad z \geq 0, \quad 0 \leq x \leq 1 \ .$$

In order to look at the boundaries at x = 0 and x = 1, Fourier transform with respect to t and z to get

$$\begin{aligned} P_{xx} &= [(\omega^2/v^2) - k_z]P \\ &= k_x P \ . \end{aligned} \tag{2}$$

It is easy to show that the solution of this equation takes the form

$$P_j \equiv P(j\,\Delta x) = A\,e^{ik_x \Delta x\, j} + B\,e^{-i k_x \Delta x\, j} \,, \tag{3}$$

where $A$ represents the amplitude of waves travelling in the +x-direction, $B$ represents waves travelling in the -x-direction, and $j \; x = x$ represents the discretization in x.

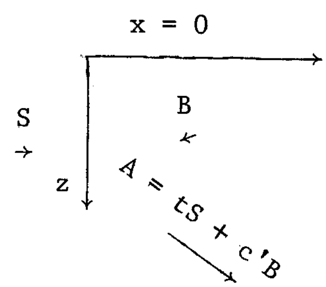
Consider now the boundary at $x = 0$. Using the Crank-Nicolson formulation of the difference equation, we can make specifications of the form

$$P_0 = EP_1 + F \qquad (4)$$

at this boundary. Inserting (3) into (4) and solving for A, we get

$$A = \frac{1}{1 - E\,e^{ik_x \Delta x}} F + \frac{E\,e^{-ik_x \Delta x} - 1}{1 - E\,e^{ik_x \Delta x}} B \, . \qquad (5)$$

By inspection, we can identify the reflection coefficient for energy from the right and the transmission coefficient for energy from the left as follows:




$$c' = (E\,e^{-ik_x \Delta x} - 1)/(1 - E\,e^{ik_x \Delta x}) \, ,$$

$$t = \beta/(1 - E\,e^{ik_x \Delta x}) \, , \qquad (6)$$

where the source-function amplitude is given by $S = F/\beta$.

The simplest inhomogeneous boundary condition we might imagine is the inhomogeneous Dirichlet condition:

$$P(x=0) = S \, ,$$

which we represent in discrete variables by

$$P_0 = S_0 \, . \qquad (7)$$

Here $\beta = 1$ and $E = 0$, so we quickly calculate

$$c' = -1 \, ,$$

$$t = 1 \, . \qquad (8)$$

The inhomogeneous form of the Neumann condition is

$$P_x(x=0) = S \, ,$$

which we may approximate by

$$(P_1 - P_0)/\Delta x = S . \qquad (9)$$

Here $\beta = -\Delta x$ and $E = 1$. Thus,

$$c' = (e^{-ik_x\Delta x} - 1)/(1 - e^{ik_x\Delta x}) = e^{-ik_x\Delta x}$$

$$= +1 + O(\Delta x) , \qquad (10a)$$

and

$$t = \frac{-\Delta x}{1 - e^{ik_x\Delta x}} = \frac{-\Delta x}{-ik_x\Delta x[1 + (k_x\Delta x/2) + \ldots]}$$

$$= (i\,m\cos\theta)^{-1} + O(\Delta x) , \qquad (10b)$$

where $m = \omega/v$, $\tan\theta = k_z/k_x$, and we assume that $\Delta x << 1$.

Comparison of (8) and (10) shows that both boundary conditions are totally reflecting, while only the Dirichlet condition transmits all source energy without amplification or attenuation. The Neumann condition introduces an amplification factor of $(i\,m\,\cos\theta)^{-1}$. The singularity for $\omega = 0$ is not as serious as it might seem since the amplification factor actually just represents the integration of the source function with respect to x. We could avoid this effect by using the boundary condition $P_x = S_x$ instead.

Next look at the inhomogeneous analog of the simple Engquist-Clayton "linear" absorbing boundary conditions:

$$[P_x - v^{-1}P_t]_{x=0} = S . \qquad (11)$$

Fourier transform with respect to t to get

$$[\hat{P}_x + im\hat{P}]_{x=0} = \hat{S} ,$$

which we approximate with

$$(1/\Delta x)(P_1 - P_0) + (im/2)(P_1 + P_0) = S . \qquad (12)$$

Here $\beta = -2\Delta x$ and $E = (2 + im\Delta x)/(2 - im\Delta x)$. After some manipulation, we get

$$c' = \frac{\cos\theta - 1}{\cos\theta + 1} + O(\Delta x) , \qquad (13a)$$

and

$$t = [-im(\cos\theta + 1)]^{-1} + O(\Delta x) . \qquad (13b)$$

Neglecting terms of $O(\Delta x)$ and higher, Eq. (13a) is the same as Engquist's reflection coefficient (SEP-10) for this boundary condition, that he derived without making the discretization assumption. The transmission coefficient for this boundary condition shows the integrative effect of the inverse of the $(\partial x + im)$ operator.

It appears that we may use any boundary condition of the form Op(P) = Op(S), where "Op" is some operator, to introduce energy into the calculation grid, and expect to get virtually the same transmission characteristics as if we had specified $P = S$ at the boundary. The choice of operator, therefore, should depend only on the reflection characteristics we desire at that boundary.

We might also use the method described above to investigate the effect of roundoff error in the boundary conditions on the interior calculations. We can rewrite the boundary condition in the form

$$Op[P(x=0)] = S + \varepsilon , \qquad (14)$$

where $\varepsilon$ represents the roundoff error. It is clear that the transmision effects on the roundoff error will be the same as those on the source, so that we can write the amplitude of the wave moving to the right as

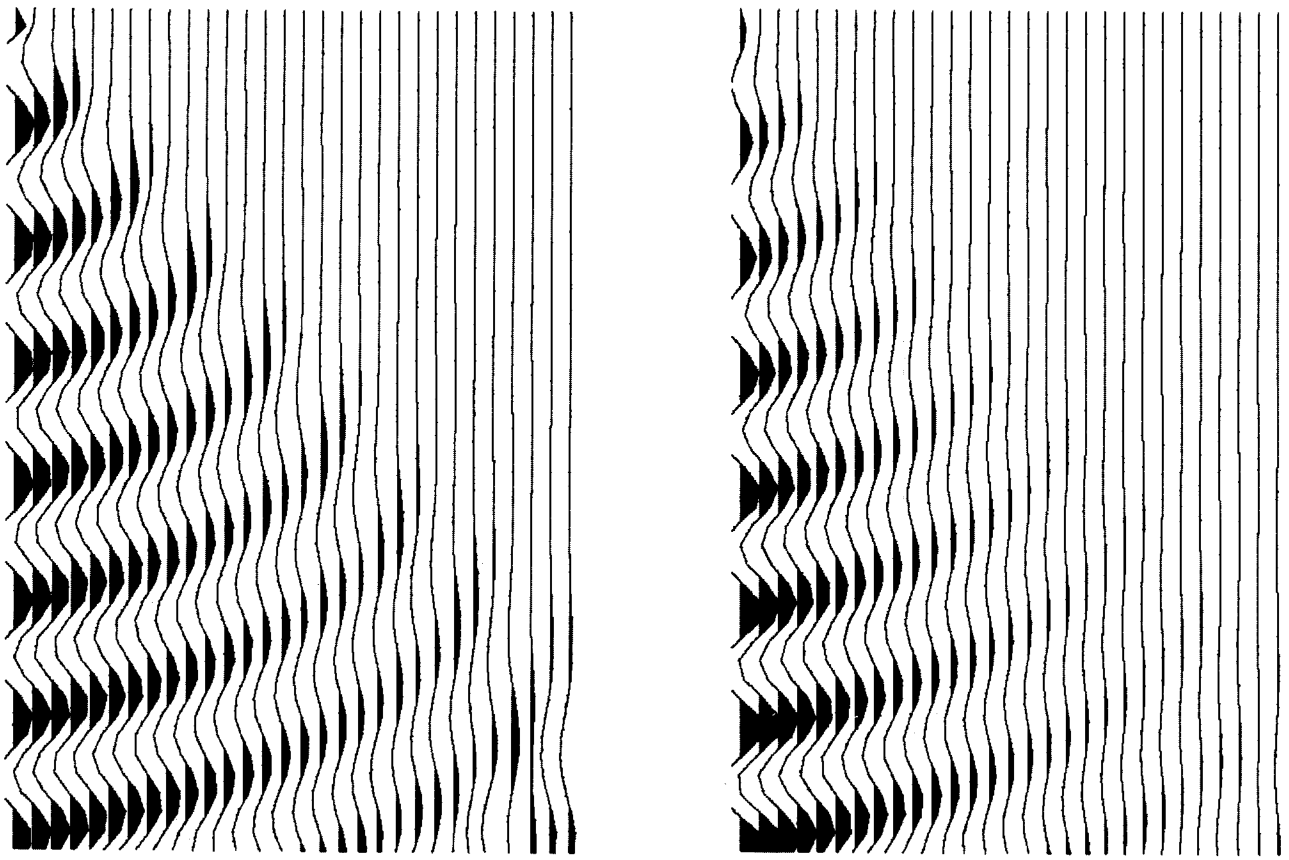
$$A = c'B + tS + t\varepsilon . \qquad (15)$$

Since roundoff noise is very high-frequency in content, boundary conditions of the forms (9) and (12) will tend to attenuate this noise, since the transmission coefficient is proportional to $\omega^{-1}$ in both cases. We therefore do not expect any stability problems to arise because of roundoff error.

## *Examples*

Several examples were run using these boundary conditions with the monochromatic wave equation in which the characteristics discussed above can be observed.

In Fig. 1, boundary conditions of $P(x=0,z) = S(z)$ [Fig. 1(a)] and $P_x(x=0,z) = S_x(z)$ [Fig. 1(b)] were used on the left-hand side. The right-hand side boundary condition was $P_x = 0$. The source is given by



(a) (b)

FIGURE 1.——In these two plots, the analytical solution for an oblique plane-wave at the left-hand boundary was used as the source function $S(z)$. In (a), the boundary condition $P(x=0,z) = S(z)$ was used, while in (b) the boundary condition $P_x(x=0,z) = S_x(z)$ was used. The results are essentially the same.

$$S(z) = e^{i m z \cos\theta} ,$$

which represents an oblique plane wave incident on the boundary. The angle $\theta$ was $\pi/12$. The results for both boundary conditions are quite similar.

Figure 2 shows the amplification effect of the boundary condition (11) on waves of two different frequencies. The source function on the left-hand boundary was the same as that used in Fig. 1. In Fig. 2(a), $m\Delta z = \pi/3$, while in Fig. 2(b), $m\Delta z = \pi/15$. We therefore expect a relative amplification in Fig. 2(b) of approximately

$$\frac{t_b}{t_a} = \frac{-i(\pi/3)(1/\Delta z)(\cos\theta + 1)}{-i(\pi/15)(1/\Delta z)(\cos\theta + 1)} = 5 ,$$

which is observed.

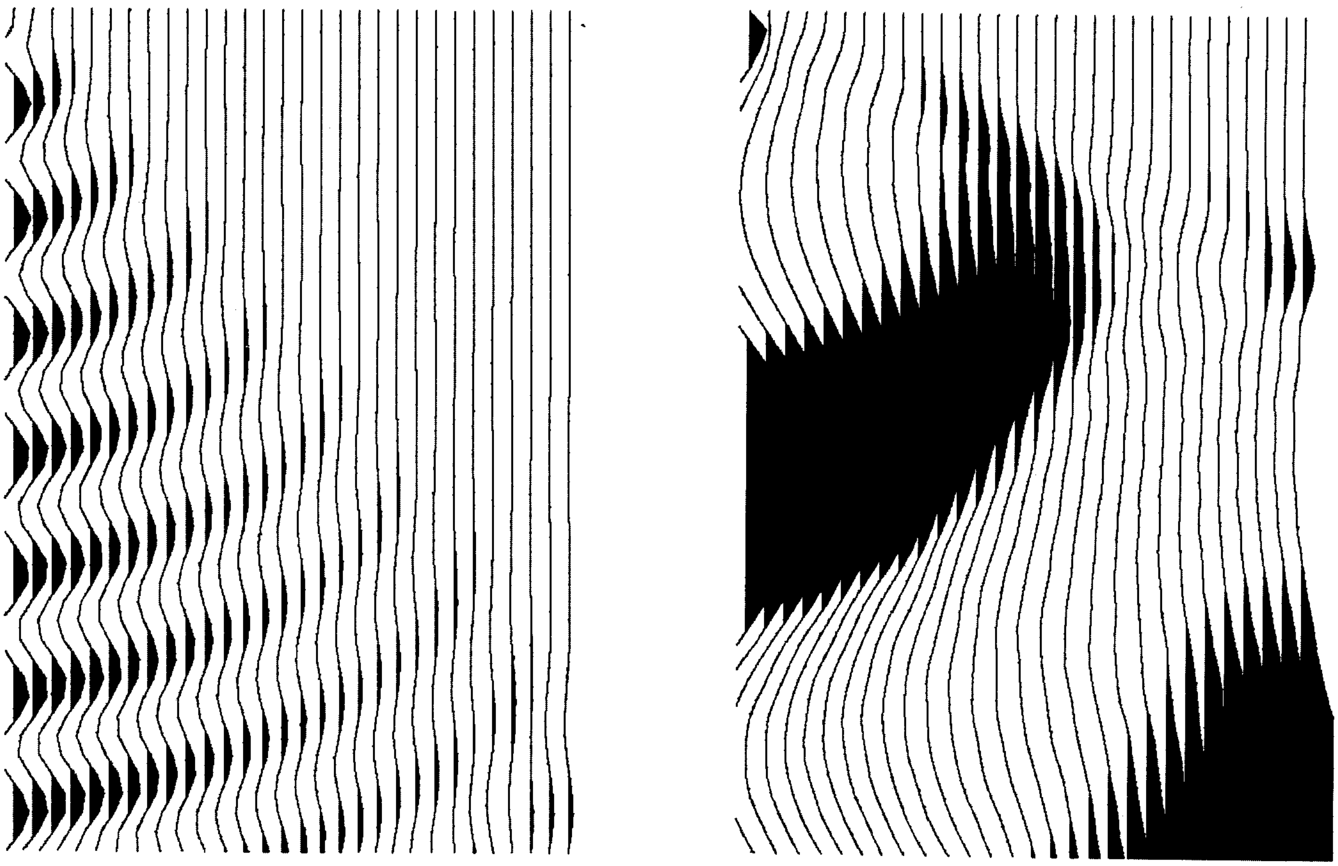
In Fig. 3, the reflection characteristics of boundary conditions (7) and (11) are compared. In both cases the source function on the left-hand boundary was of the form

$$S(z) = e^{-\alpha z} e^{imz \cos\theta} ,$$

so that the source dies out exponentially in $z$ . Zero-slope boundary conditions are posed on the right-hand side. In Fig. 3(a), boundary condition (7) allows source energy to be transmitted, but reflects internal energy striking the boundary. In Fig. 3(b), boundary condition (11) allows source energy to be transmitted and absorbs internal energy that strikes the boundary. The monochromatic wave equation was used, but the solution for three frequencies was summed.

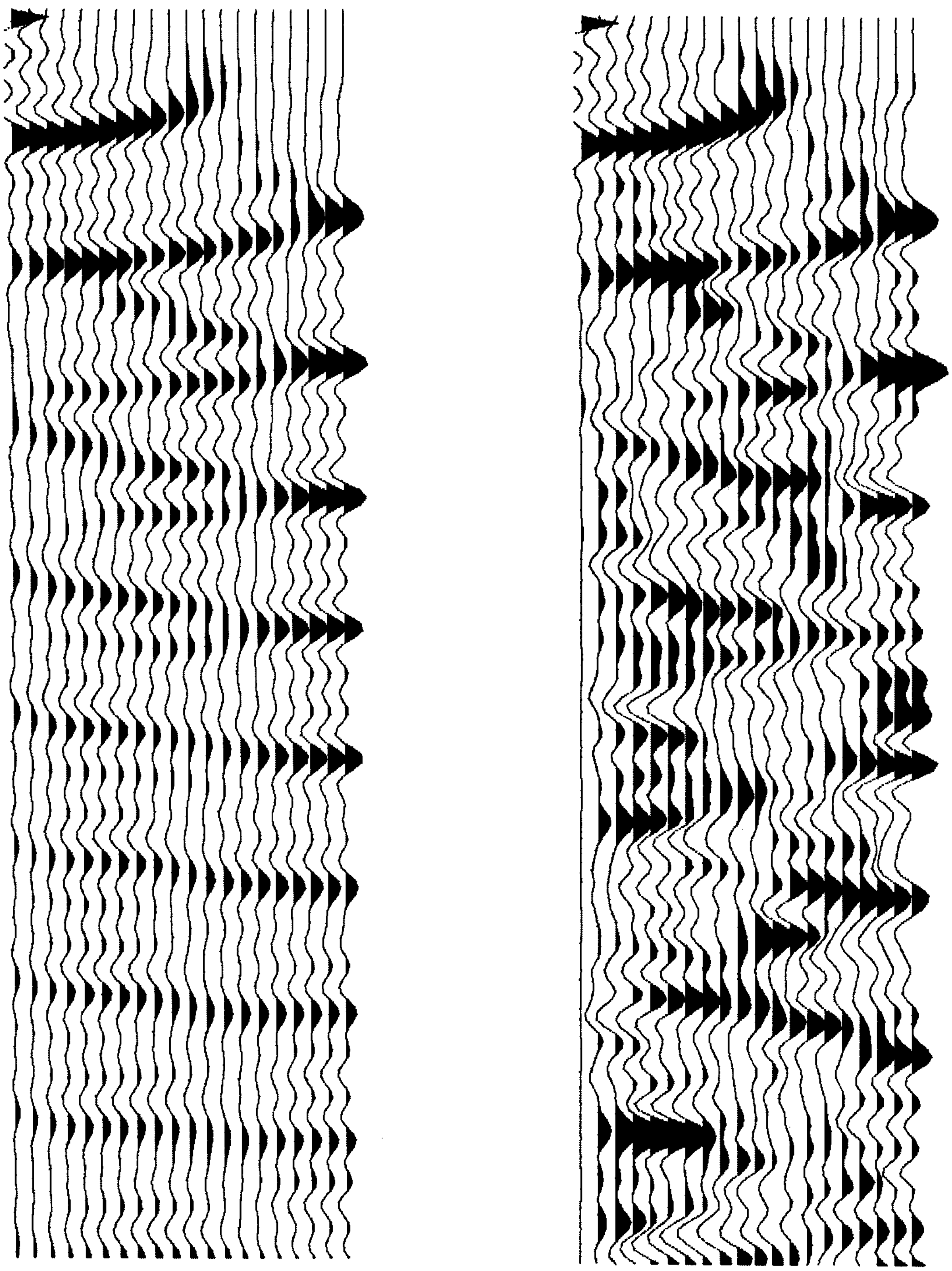
## *Conclusions*

We have seen that it is relatively simple to introduce energy into a wave-equation calculation from the side boundaries. The transmission and reflection properties of the boundaries have been derived for three simple inhomogeneous boundary conditions, and numerical examples are given that demonstrate the predicted behavior.



(a) (b)

FIGURE 2.——The inhomogeneous absorbing boundary condition, Eq. (11), was used on the left-hand boundary for both these plots. The source function S was an oblique plane wave of a single frequency. The frequency used in (a) was 5 times that used in (b). The $\omega^{-1}$ amplification effect of the transmission coefficient, Eq. (13b), is observed.



(a) (b)

FIGURE 3.——In these two plots, boundary conditions of the form Op[P(x=0,z)] = Op[S(z)] were used at the left-hand boundary, while reflective ($P_x$ = 0) boundary conditions were used at the right-hand boundary. In (a) the "absorbing" operator was used, while in (b) a reflecting operator was used. As the source function decays, mostly energy propagating to the left is seen in (a), while in (b) the energy is "trapped by the reflecting sides. Scale of plots is about 2:5.