A Paraxial Equation for Elastic Waves

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In this paper we shall present a method for putting the elastic wave equations in a form that is stable under depth extrapolation. This paper represents only the preliminary work on the method as no numerical computations have been done with the results.

The starting point of the derivation is the matrix formulation of the elastic wave equation in terms of displacements and stresses, given by Claerbout (FGDP*, equation (9-6-1), p. 182). Since equation (9-6-1) lacks a derivation in FGDP we will give one here. We start with Hooke's Law for isotropic bodies

$$\tau_{ij} = \lambda \operatorname{div} \underline{u}\, \delta_{ij} + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$

If we consider the two dimensional case and define the components of $\underline{u}$ as (u,w) and of $\underline{x}$ as (x,z), then the stress-strain relations become

$$\tau_{xx} = (\lambda + 2\mu)\, u_x + \lambda\, w_z \tag{1}$$

$$\tau_{zz} = (\lambda + 2\mu)\, w_z + \lambda\, u_z \tag{2}$$

$$\tau_{zx} = \tau_{xz} = \mu\,(u_z + w_x) \tag{3}$$

To obtain the elastic equations we now employ Newton's Law relating acceleration and force which in this case is the elastic force $\operatorname{div} \tau$. Hence,

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$$\rho\, u_{tt} \;=\; \partial_x\, \tau_{xx} + \partial_z\, \tau_{zx} \tag{4}$$

$$\rho\, w_{tt} \;=\; \partial_x\, \tau_{zx} + \partial_z\, \tau_{zz} \tag{5}$$

We can put equations (1)-(5) in a convenient state variable form

$$\frac{\partial}{\partial z}\begin{pmatrix} \underline{r} \\ \\ \underline{s} \end{pmatrix} = \begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix}\begin{pmatrix} \underline{r} \\ \\ \underline{s} \end{pmatrix} \tag{6}$$

if we choose $\underline{r}$ to be (u,τ_{zz}) and $\underline{s}$ to be (w,τ_{xz}) .

From (3) $$u_z \;=\; \frac{1}{\mu}\,\tau_{xz} - \partial_x\, w_x \tag{7}$$

From (5) $$\partial_z\, \tau_{zz} \;=\; \rho\, w_{tt} - \partial_x\, \tau_{zx} \tag{8}$$

From (2) $$w_z \;=\; \frac{1}{\lambda+2\mu}\,\tau_{zz} - \frac{\lambda}{\lambda+2\mu}\,\partial_x\, u \tag{9}$$

From (4) $$\partial_z\, \tau_{xz} \;=\; \rho\, u_{tt} - \partial_x\, \tau_{xx}$$

τ_{xx} is not a state variable and must be eliminated from the last equation. Using (1)

$$\partial_z\, \tau_{xz} \;=\; \rho\, u_{tt} - \partial_x\,[\,(\lambda+2\mu)\,\partial_x u + \lambda\;\partial_z\, w\,]$$

To eliminate the z derivative from the RHS we use (9). Hence,

$$\partial_z\, \tau_{xz} \;=\; \rho\, u_{tt} - \partial_x\,[\,(\lambda+2\mu)\,\partial_x u + \lambda\,(\frac{1}{\lambda+2\mu}\,\tau_{zz} - \frac{\lambda}{\lambda+2\mu}\,\partial_x\, u)\,]$$

$$= \rho\, u_{tt} - \partial_x\,(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu})\,\partial_x\, u \;-\; \partial_x\,(\frac{\lambda}{\lambda+2\mu})\,\tau_{zz} \tag{10}$$

Combining equations (7)-(10) into the matrix form of (6) we find that T has a block form

$$\frac{\partial}{\partial z}\begin{pmatrix}\underline{r}\\ \underline{s}\end{pmatrix} = \begin{pmatrix}0 & B\\ A & 0\end{pmatrix}\begin{pmatrix}\underline{r}\\ \underline{s}\end{pmatrix}$$

where

$$A = \begin{pmatrix} \frac{-\lambda\,\partial_x}{(\lambda+2\mu)} & \frac{1}{\lambda+2\mu} \\ \rho\partial_{tt}-\gamma & \frac{-\partial_x\,\lambda}{\lambda+2\mu} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\partial_x & \frac{1}{\mu} \\ \rho\partial_{tt} & -\partial_x \end{pmatrix}$$

where

$$\gamma = \partial_x \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \partial_x$$

If a time dependence of $e^{-i\omega t}$ is assumed then ∂_{tt} terms may be replaced by $-\omega^2$ which leads to equation (9-6-1).

Two facts are worth noting about the above derivation. First, the matrix elements are operators, not simple coefficients. Secondly, homogeneity was not assumed in the derivation of (9-6-1). However, we will make this assumption in the remainder of the paper.

Equation (11) is first order in z and could be used for depth extrapolation if it weren't for the fact that the exponentially increasing evanescent waves would soon dominate the solution.

To obtain a formulation that removes the evanescent component we proceed as follows

$$\frac{\partial}{\partial z}\underline{s} = A\,\underline{r} \qquad \frac{\partial}{\partial z}\underline{r} = B\,\underline{s}$$

$$\frac{\partial^2}{\partial z^2}\underline{s} = A\frac{\partial}{\partial z}\underline{r} = A\cdot B\,\underline{s}$$

Fourier transforming in z and x we have

$$k_z^2 \, S = A \cdot B \; S$$

or

$$k_z = \sqrt{A \cdot B} \tag{12}$$

where $A \cdot B$ is a function of k_x .

If the solution is assumed to have a z dependence of the form $e^{i k_z z}$, then the evenescent components will be absent if k_z can be made to be strictly real. This is done by expanding the square root of (12) in even powers of k_x .

However, problems immediately arise because the product $A \cdot B$ has odd order terms in k_x, which cannot be neglected as being small. (For example, the product $(a_{21} \; b_{11} + a_{22} \; b_{21})$ contains the term $\rho\omega^2 (1 + \frac{\lambda}{\lambda+2\mu} (i k_x))$. This means that a straightforward Taylor Series expansion in powers of k_x^2 will not work.

To overcome this problem we sought to modify the original state variables of (11) in a way that odd-order derivatives are avoided. This was accomplished with the following changes in $\underline{r}$ and $\underline{s}$.

$$\underline{r} = \begin{pmatrix} u \\ \partial_x \tau_{zz} \end{pmatrix} \qquad \underline{s} = \begin{pmatrix} \partial_x w \\ \tau_{zx} \end{pmatrix}$$

which necessitates that A and B be changed to

$$A = \begin{pmatrix} \frac{-\lambda\,\partial_x^2}{\lambda+2\mu} & \frac{1}{\lambda+2\mu} \\ -\rho\omega^2-\lambda & \frac{-\lambda}{\lambda+2\mu} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & \frac{1}{\mu} \\ -\rho\omega^2 & -\partial_x^2 \end{pmatrix}$$

The product $A \cdot B$ now has the form

$$\underbrace{-\omega^2 \begin{pmatrix} \frac{\rho}{\lambda+2\mu} & 0 \\ \frac{-2\rho(\lambda+\mu)}{\lambda+2\mu} & \frac{\rho}{\mu} \end{pmatrix}}_{L} \; \underbrace{-k_x^2 \frac{1}{\lambda+2\mu} \begin{pmatrix} \lambda & \frac{-(\lambda+\mu)}{\mu} \\ 4\mu(\lambda+\mu) & -4(\lambda+\mu)+\lambda \end{pmatrix}}_{F}$$

Now we have the relation

$$k_z = (L + k_x^2 F)^{1/2}$$

To obtain a 2^{nd} order approximation to the square root we note that

$$L + k_x^2 F = (L^{1/2} + k_x^2 C)^2 - \underbrace{k_x^2 C^2}_{\text{error term}} \tag{13}$$

The dispersion for the modified elastic wave equation is then

$$k_z = L^{1/2} + k_x^2 C \tag{14}$$

If L is denoted symbolically as

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

then

$$L^{1/2} = \begin{pmatrix} a^{1/2} & 0 \\ \frac{b^{1/2}}{a^{1/2}+c^{1/2}} & c^{1/2} \end{pmatrix} \equiv \begin{pmatrix} \ell_1 & 0 \\ \ell_2 & \ell_3 \end{pmatrix}$$

To find C we equate the coefficients of k_x^2 on each side of (13). Thus,

$$(C L^{1/2} + L^{1/2} C) = F$$

The solution for C is a 4x4 system which can be solved relatively easily

$$\begin{pmatrix} 2\ell_1 & \ell_2 & 0 & 0 \\ 0 & \ell_1+\ell_3 & 0 & 0 \\ \ell_2 & 0 & \ell_1+\ell_3 & \ell_2 \\ 0 & \ell_2 & 0 & 2\ell_3 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \end{pmatrix}$$

The solution for the other half of the state variables in (11) follows along the same lines as $\underline{s}$ except that the 'L' matrix is upper triangular.