# Exponential Tilt Invariant Solution of Overdetermined Convolution Equations

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In reflection seismology we have frequent occasion to work with over-determined simultaneous equations of the type

$$\begin{bmatrix} x_0 & 0 & 0 \\ x_1 & x_0 & 0 \\ x_2 & x_1 & x_0 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & x_2 \\ 0 & x_4 & x_3 \\ 0 & 0 & x_4 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \qquad (1)$$

Although such work appears to be precise and mathematically rigorous, there are at the outset some subjective decisions whose choice predetermines the behavior of the result. One such choice is the choice between least squares and $L_1$ . Another is the choice of weighting functions. The theory is well developed for $x_t$ and $y_t$ being stationary Gaussian random time series where unity weighting functions are natural. But in practice the $x_t$ and $y_t$ are usually seismograms. Qualitatively the mismatch between theory and practice can be eased by means of weighting functions. But the solution obtained is a first order function of the weights, and there is no very solid means of determining what weights are appropriate. Another problem is that weights destroy the Toeplitz character of (1) unless they are applied to the raw data $x_t$ and $y_t$. This of course amounts to falsification of data points, something we may wish to avoid, particularly with bright spots and amplitude relations of multiple reflections. The loss of Toeplitz character imposes much higher computational effort.

A particular weighting function, the exponential $e^{\alpha t}$ offers some hope for preserving Toeplitz character. Applying the weight $e^{\alpha t}$ to the t-th equation in (1) alters the set so that columns are no longer shifted replicas of one another. But if we define new variables $f^{(\alpha)}$ where

$$f_t^{(\alpha)} = f_t \, e^{\alpha t} \qquad (2a)$$

then in terms of the new variables we retain a Toeplitz set. To see this, let us also define

$$\varepsilon_t^{(\alpha)} = \varepsilon_t \, e^{\alpha t} \qquad (2b)$$

$$y_t^{(\alpha)} = y_t \, e^{\alpha t} \qquad (2c)$$

$$x_t^{(\alpha)} = x_t \, e^{\alpha t} \qquad (2d)$$

Solving (2) for $f_t$, $\varepsilon_t$, $y_t$, and $x_t$ and substituting into (1), then multiplying the t - th equation by $e^{\alpha t}$ gives a set of equations for $f^{(\alpha)}$ which has Toeplitz character, that is, the columns $x_t^{(\alpha)}$ are down shifted replicas of one another.

In any particular application we choose a value for $\alpha$ and a numerical technique for solution to (1). We might minimize the sum of $|\varepsilon_t^{(\alpha)}|$ or the sum of the squares. We might use Weiner-Levinson end effect treatment or we might use Burg end effect treatment. For stationary time series the parameter $\alpha$ would be chosen to be zero. Choice of a positive $\alpha$ makes the bottom equations fit better and negative $\alpha$ makes the top equations fit better. For reflection seismograms where amplitudes decrease with time but information is hardly related to amplitude, we might choose a positive $\alpha$ , hopefully to distribute the error residuals evenly along the time axis as with stationary time series.

I will now define the exponential tilt invariance property of overdetermined simultaneous equation solving methods. A method of solving $x^{(\alpha)} * f^{(\alpha)} \approx d^{(\alpha)}$ for $f^{(\alpha)}$ will be said to be exponentially tilt invariant if and only if for all $\alpha_1$ , $\alpha_2$ , and $t$

$$f_t^{(\alpha_2)} e^{-\alpha_2 t} = f_t^{(\alpha_1)} e^{-\alpha_1 t} \tag{3}$$

It will be shown that all least squares solvers do not have the exponential tilt invariant property and a tilt invariant solver will be proposed. Why are we interested in a tilt invariant technique? To begin with, it means we no longer need consider the problem of determining an optimum value for $\alpha$ . Obviously there will exist many data sets for which no choice of the single number $\alpha$ will distribute the residuals nicely along the time axis. Furthermore, the class of all exponential functions is pretty big, so that if an exponential tilt invariant technique can be found then we are entitled to hope that such a technique would be insensitive to any smoothly variable weighting function. This of course is just what we want in reflection seismology where the information carried by a signal is largely independent of a slowly variable magnitude function. This is not to say that data points can be falsified by a slowly variable gain function. Such falsification might completely destroy our ability to predict deep water multiples. The distinction is a subtle one, namely that the geological significance of an echo is independent of a slowly variable magnitude, but the data itself should not be falsified by a slowly variable magnitude.

Now, let us see why least squares does not give tilt invariant results. Look at the simple case where the filter $f_t$ reduces to just one point $f_0$ . Then the answer is

$$f_0^{(\alpha)} = \frac{\Sigma\, x_t^{(\alpha)}\, y_t^{(\alpha)}}{\Sigma\, x_t^{(\alpha)}\, x_t^{(\alpha)}} = \frac{\Sigma\, x_t\, y_t\, e^{-2\alpha t}}{\Sigma\, x_t^2\, e^{-2\alpha t}}$$

which means that $f_0^{(\alpha)}$ comes out different for different $\alpha$ which means that

$$f_0^{(\alpha_1)}\, e^{\alpha_1 0} \neq f_0^{(\alpha_2)}\, e^{\alpha_2 0}$$

On the other hand, suppose $f_0^{(\alpha)}$ was determined by

$$f_0^{(\alpha)} = \underset{t}{\text{Median}}\, \frac{y_t^{(\alpha)}}{x_t^{(\alpha)}} = \underset{t}{\text{Median}}\, \frac{y_t\, e^{\alpha t}}{x_t\, e^{\alpha t}}$$

This answer is obviously exponentially tilt invariant. Likewise, consider the solution where all $f_t$ are taken zero except for $f_1$ . Then we also deduce tilt invariance as follows

$$f_1^{(\alpha)} = \underset{t}{\text{Median}} \frac{y_{t+1} e^{\alpha(t+1)}}{x_t \; e^{\alpha t}} = e^{\alpha} \text{ Median } \frac{y_{t+1}}{x_t} = e^{\alpha} f_1^{(0)}$$

Insensitivity to <u>all</u> weighting functions is a property we associate with the mathematical fact that the following two conditions (from "Non-Gaussian Signal Analysis", (10a,b), p. 83) are equivalent.

$$\{ 0 = \text{Median} ( y_i / x_i ) \} \Longleftrightarrow \{ 0 = \sum_i \text{ sgn} ( y_i ) \text{ sgn} ( x_i ) \}$$

In the next section I am planning to show in detail how Levinson or Burg recursions can be made tilt invariant. However, it is easy to indicate the main idea. In multichannel Levinson recursion the reflection coefficient for forward prediction does not equal that for backward prediction. In order to obtain the tilt invariant property for single channel time series we must consider these reflection coefficients also to have the possibility of being different, even though they are forced to be the same in the usual single channel problem. We will determine the reflection coefficients

$$c^+ = \text{Median} \frac{e_t^+}{e_t^-}$$

$$c^- = \text{Median} \frac{e_t^-}{e_t^+}$$

It may turn out that either $c^+$ or $c^-$ is greater in magnitude than unity, but it is easily shown that the product $c^+c^-$ is less than or equal to plus one. The case $c^+c^- = 1$ arises only if there is sign agreement within each pair $(e_t^+, e_t^-)$. What is nice, of course, is that in the event of stationary Gaussian random variables the expected value of both $c^+$ and $c^-$ is the usual Levinson reflection coefficient.