Absorptive Side Boundaries for Migration

by Jon F. Claerbout

In the past I have used mostly zero slope side boundary conditions. Bjorn Engquist recently told us of an improved boundary condition depicted in Figure 1.

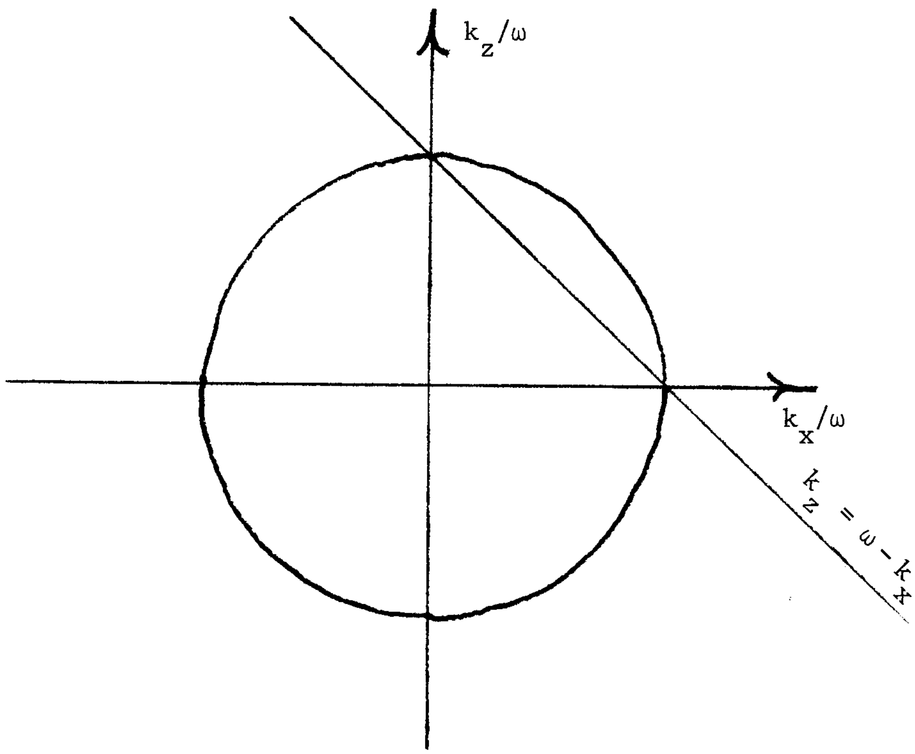

Figure 1. Engquist's absorptive side boundary condition.

Bjorn's side boundary condition is found by expressing the dispersion relation $k_z = \omega - k_x$ in the physical domain as $P_z = - P_t - P_x$ or, in the computational domain as $P'_{z'} + P'_{x'} = 0$. (For the other side of the mesh one would use $P'_{z'} - P'_{x'} = 0$.)

We will later see that the reason his boundary condition absorbs so nicely is that his dispersion relation is a better approximation to a quadrant of a circle than $k_x = 0$, the dispersion relation of the usual zero slope side condition $P_x = 0$.

Thus, intending to fit a quarter circle, I propose to consider equation (1)

$$y_{edge} = \frac{a\,(x_0 - x)}{a\,x_0 - x} \qquad (1)$$

Letting $y_{edge} = k_z/\omega$ and $x = k_x/\omega$, equation (1) becomes the dispersion relation

$$\left(a\,x_0 - \frac{k_x}{\omega} \right) \frac{k_z}{\omega} - a \left(x_0 - \frac{k_x}{\omega} \right) = 0 \qquad (2a)$$

or

$$D\,(k_x, k_z) = [\,(a\,x_0\,\omega - k_x)\,k_z - a\,(\omega^2\,x_0 - \omega\,k_x)\,] = 0 \qquad (2b)$$

which inverse transforms to the boundary condition

$$[\,(a\,x_0\,\partial_t + \partial_x)\,\partial_z + a\,(x_0\,\partial_{tt} + \partial_{xt})\,]\,P = 0 \qquad (3)$$

This is easily expressed at two adjacent points on the x axis, on the right hand edge of the mesh. Equation (1) was the most general expression I could think of for circle fitting under the restraint that (3) not involve ∂_{xx} or higher order terms.

Now let us return to the matter of numerical choice of x_0 and a . We could choose a and x_0 to make a best fit to a quarter circle, but we are usually more interested in matching the parabolic dispersion relation

$$y_{interior} = 1 - x^2/2 \tag{4a}$$

or

$$k_z = \omega - k_x^2 / (2\omega) \tag{4b}$$

I propose to define optimum by matching the boundary dispersion curve (1) to the interior dispersion curve (4) at three points. Note that at $x = 0$ we already have $y_{edge} = y_{interior}$. Likewise, $y_{edge} = 0$ implies $x = x_0$ in (1) and $y_{interior} = 0$ implies $x = \sqrt{2}$ in (4) . Thus, we take $x_0 = \sqrt{2}$. Finally, pick another point x_1 and define y_1 from (4a) as $y_1 = 1 - x_1^2/2$. Then, inserting (x_1, y_1) into (1) we may solve for a obtaining

$$(a x_0 - x_1) y_1 = a(x_0 - x_1)$$

$$a(x_0 y_1 + x_1 - x_0) = x_1 y_1$$

$$a = \frac{x_1 y_1}{x_0 y_1 + x_1 - x_0} \tag{5}$$

For a perfect fit at a 30° angle $x_1 = .5$, $y_1 = .875$ and

$$a = 1.353553389$$

Last, it is a simple matter to deduce the reflection coefficient at the side boundary. This will show why we are fitting the boundary dispersion relation to one side of the interior dispersion relation. Consider a unit amplitude incident wave field and a side reflected wave field amplitude c given by

$$p = e^{-i\omega t + ik_z z} \left(e^{ik_x x} + c\, e^{-ik_x x} \right) \tag{6}$$

In (6) $k_z(k_x)$ is defined by the interior dispersion relation (4b). Inserting (6) into the boundary condition (3) and noting the definition of $D(k_x, k_z)$ in (2b) we will get

$$D(k_x, k_z) + c\, D(-k_x, k_z) = 0 \tag{7}$$

which we may solve for the reflection coefficient c as

$$c = \frac{-D(k_x, k_z)}{D(-k_x, k_z)} \tag{8}$$

It is important to realize that the k_z occurring in (7) and (8) is that of $y_{interior}$. If $y_{interior}$ happens to equal y_{edge}, then D vanishes implying that the reflection coefficient vanishes. That is why we set out to match a quarter circle or single side of a parabola. A simple program and table of reflection coefficientsis given in Figure 2. It will be noted that at grazing incidence the reflection coefficient is -1. At $\sin 11.5° = .2$ the reflection coefficient is $-.32$ and rapidly diminishing so that everywhere beyond 23° the reflection coefficient is less than .07. The large reflection coefficient near grazing incidence is not especially harmful since such reflected rays move slowly back into the mesh.

```
$NATURV
          X0=SORT(2.)
          X1=SIN(2.*3.14159*30./360.)
          Y1=1-X1*X1/2.
          A=X1*Y1/(X0*Y1+X1-X0)
          PRINT,A
          DO 10 IX=1,16
          X=(IX-1)/10.+.001
          YINT=1.-X*X/2.
          YEDGE=A*(X0-X)/(A*X0-X)
          DPLUS=(A*X0-X)*YINT-A*(X0-X)
          DMINUS=(A*X0+X)*YINT-A*(X0+X)
          C=-DPLUS/DMINUS
          PRINT,X,YINT,YEDGE,C
   10     CONTINUE
          STOP
          END
```

			Reflection Coefficient
0.1353557E 01			
0.9999999E-03	0.9999995E 00	0.9998150E 00	0.9946236E 00
0.1010000E 00	0.9948995E 00	0.9803060E 00	0.5753805E 00
0.2010000E 00	0.9797995E 00	0.9585196E 00	0.3203794E 00
0.3009999E 00	0.9546996E 00	0.9340321E 00	0.1612434E 00
0.4010000E 00	0.9195995E 00	0.9063081E 00	0.6133197E-01
0.5010000E 00	0.8744996E 00	0.8746606E 00	-0.4746101E-03
0.6010000E 00	0.8193995E 00	0.8381940E 00	-0.3701918E-01
0.7010000E 00	0.7542995E 00	0.7957147E 00	-0.5643017E-01
0.8009999E 00	0.6791996E 00	0.7456042E 00	-0.6404436E-01
0.9010000E 00	0.5940996E 00	0.6856015E 00	-0.6344646E-01
0.1000999E 01	0.4990001E 00	0.6124583E 00	-0.5710464E-01
0.1100999E 01	0.3939009E 00	0.5213279E 00	-0.4674505E-01
0.1200999E 01	0.2788005E 00	0.4046409E 00	-0.3359333E-01
0.1300999E 01	0.1537013E 00	0.2498932E 00	-0.1354435E-01
0.1400999E 01	0.1860094E-01	0.3485078E-01	-0.2224536E-02
0.1500999E 01	-0.1264992E 00	-0.2842814E 00	0.1439252E-01

Figure 2. Program and result showing boundary dispersion relation and reflection coefficient.

This analysis has ignored the effects of sampling. Because present day recording techniques push us so close to aliasing on the x axis, a more complete analysis would seem to be worthwhile.