A Brief Derivation of the Dix Theorem

by Jon Claerbout

The equation for a circle in (x,z) space or a hyperbola in (x,t) space is

$$x^2 + z^2 = \bar{v}^2 t^2 \tag{1}$$

The question is this: Suppose waves propagate outwards from a surface point source in a stratified medium. The reflected waves are then fit, by some procedure, to a hyperbola and a $\bar{v}$ is determined. In what sense does $\bar{v}$ represent an average of the velocities in the layers? Begin by differentiating (1) with respect to x at a constant value of z.

Thus,

$$2x = \bar{v}^2 \, 2t \left(\frac{\partial t}{\partial x}\right)_z$$

or

$$\bar{v}^2 = \frac{x}{t} \left(\frac{\partial t}{\partial x}\right)_z^{-1} \tag{2}$$

From Figure 1 we see that in any layer the sine of the ray angle from the vertical is given by

$$\sin\theta = \frac{v \, dt}{dx}$$

which we can recognize as

$$\left(\frac{\partial t}{\partial x}\right)_z = \frac{\sin\theta(z)}{v(z)} = p \neq p(z) \tag{3}$$

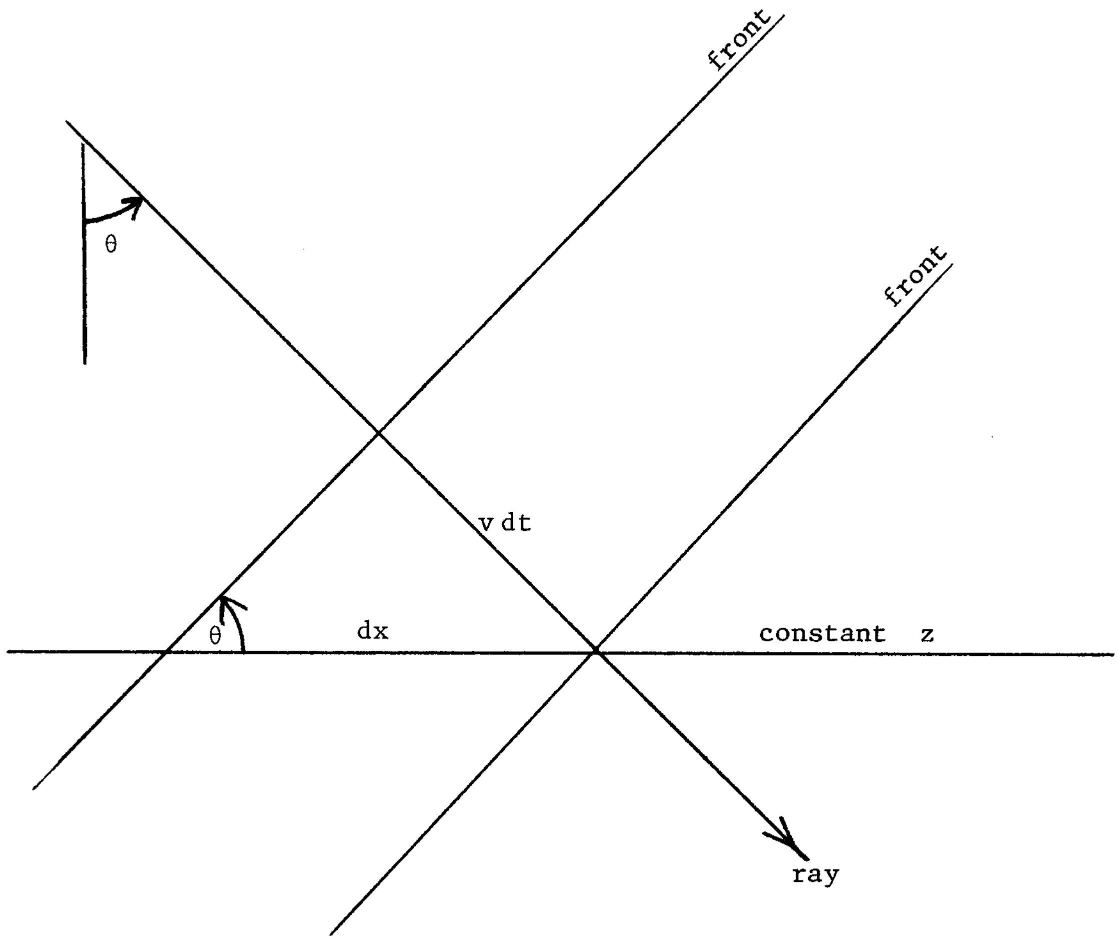

Figure 1. Diagram to illustrate that

$$p = \frac{\sin\theta}{v} = \left(\frac{\partial t}{\partial x} \right)_z$$

We wish to develop a mapping between the (x,z) coordinates in the vicinity of the source and (p,t) coordinates defined as ray parameter and travel times of rays leaving the origin $(x,z) = (0,0)$. Let the velocity $v(x,z) = v(z)$ be representable in either frame

$$v(x,z) = v(z) = v'(p,t)$$

Now the x coordinate of the tip of the ray as a function of (p,t) will be

$$\begin{aligned} x(p,t) &= \int_0^t v'(p,t) \sin\theta(p,t) \, dt \\ &= p \int_0^t v'(p,t)^2 \, dt \qquad (4) \end{aligned}$$

Inserting (3) and (4) into (2) we have

$$\bar{v}^2 = \frac{1}{t} \int_0^t v'(p,t)^2 \, dt \qquad (5)$$

This shows that $\bar{v}$ is the root-mean-square velocity of the wave along its path. Notice that the "straight ray" approximation which occurs in some derivations is not really necessary. Of course the data will not be an exact hyperbola. But, if it is windowed about some particular x and t and $\partial t / \partial x$ is measured in that window then the $\bar{v}$ determined in that window will be exactly the RMS velocity for that particular ray (p).

From study of Figure 2, you should be able to recognize that

$$t'(p,d) = 2\tau(p,d) - 2\,p\,x(p,d) \tag{11}$$

Using (9) and (10) we find

$$t'(p,d) = 2\int_0^d \left(\frac{1}{v\cos\theta} - p\tan\theta \right) dz$$

$$t'(p,d) = 2p\int_0^d \left(\frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} \right) dz$$

$$t'(p,d) = 2p\int_0^d \frac{\cos\theta}{\sin\theta}\, dz \tag{12}$$

Inverse interpolation enables us to construct a table $d(p,t')$ from (12). The lateral shift of interest is

$$\Delta x(p,t') = \int_0^{d(p,t')} \tan[\theta(z)]\, dd \tag{13a}$$

$$= \int_0^{t'} \tan(\theta)\, \frac{dd}{dt'}\, dt' \tag{13b}$$

Differentiating (12) gives dt'/dd which reduces (13b) to Schultz's result

$$\Delta x(p,t') = \frac{1}{2p}\int_0^{t'} \tan^2[\,\theta(p,t')\,]\, dt' \tag{14}$$

Equation (14) is what you need when you have downward continued data in (x',t') space and you wish to assume a velocity so that you can laterally shift the data in order to display it in (x,t') space.

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(revised)

Slant Plane Wave Interpretation Coordinates

by Philip S. Schultz

Although field data is collected in common shot gathers, this format as a data display for interpretation is unsatisfactory because within a gather the energy from a given horizontal reflector does not emanate from a unique subsurface reflecting point. A more desirable data display is the CDP gather for which, in a non-dipping earth, any given event on a gather has the same subsurface reflection point.

We now face a similar situation in our slant frames. The slant stacks, having been done over common shot and common geophone gathers, exhibit the similar tendency for the subsurface reflection point to roam. Figure 1 shows this phenomenon for a single trace in a constant velocity medium. Although the upcoming ray paths of all three reflectors are colinear, they clearly do not represent energy emanating from reflection points directly below the geophone.

If we had a seismic section composed of many traces all of the same propagation angle, θ , we would (for interpretation purposes at least) like to perform a coordinate transformation resulting in a uniform shearing action on these data, so that we might place reflected energy in its proper horizontal position.

Our starting point will be the horizontal and vertical coordinates which result from the stacking process itself (x' , t') and a perfect knowledge of the depth-dependent rms velocity , $v_{rms}(z)$.

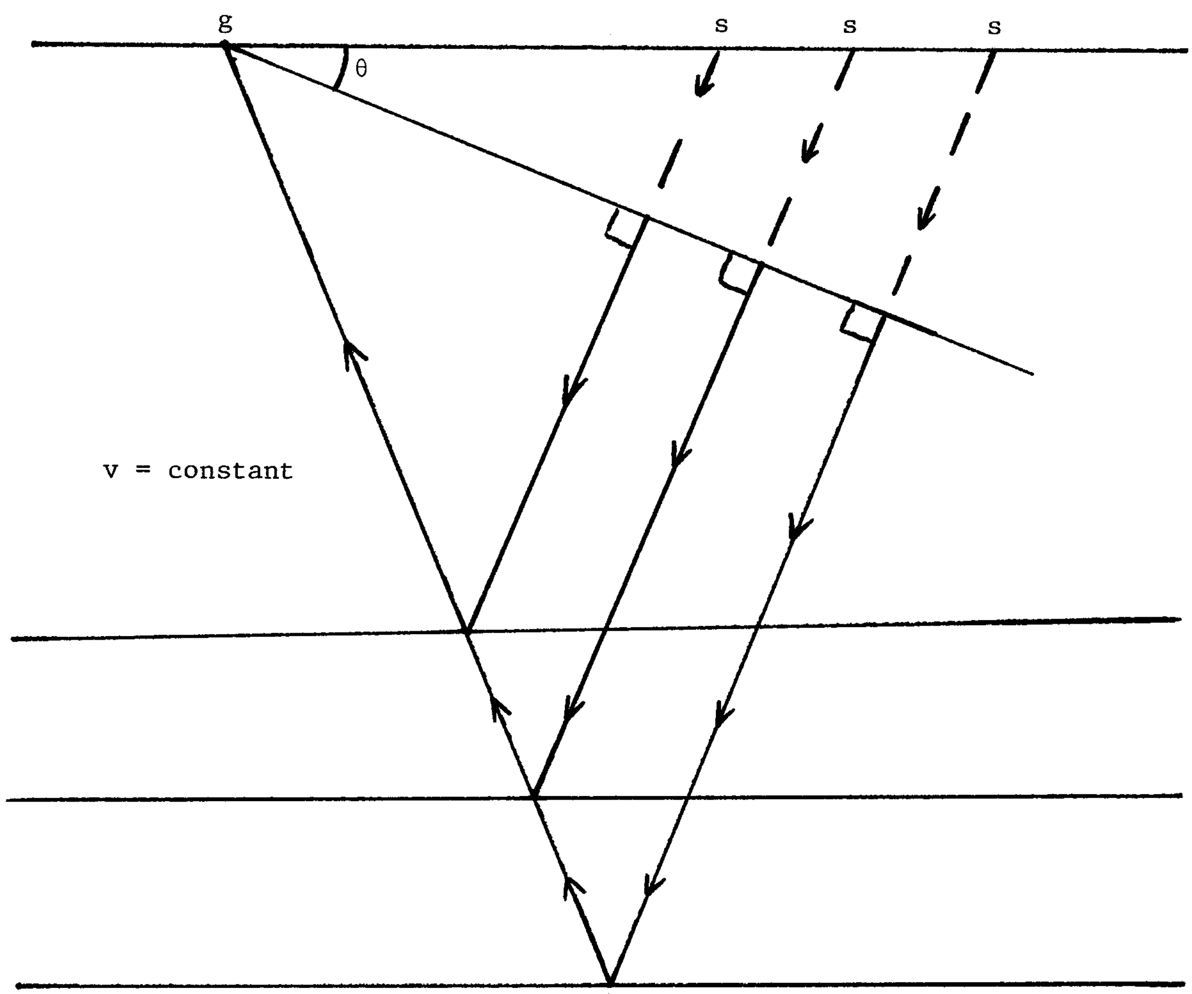

Figure 1. Ray paths in the slant frame for three horizontal reflectors in a constant velocity medium. The energy received at the geophone is displayed as a single trace by the slant stack process, but as seen in the Figure really represents subsurface reflection points which lie along a ray path.

Figure 2 shows a ray path diagram for energy reaching a geophone from a buried reflector. The ray path is that expected from data which have been slant wave stacked over common geophone gathers. The upcoming ray has ray parameter, p , and intercepts the geophone position. Through a given horizontally stratified earth medium, these two conditions uniquely define a ray path. After a slant plane wave stack over ray parameter p , this ray path (actually a Fresnel region around the ray path) is the only one through which energy can be transmitted to geophone position, g . If we define x' to be the horizontal coordinate of a trace that we obtain from a slant plane stack, then x' is constant over the ray path.

The transformation we seek, therefore, is to the coordinate labeled x in the Figure. This is the actual horizontal coordinate of the sub-surface reflection point, and is related to x' and z_0 by

$$x = x' + f(z_0) / 2$$

$$= x' + \int_0^{z_0} \tan\theta(z)\, dz \qquad (1)$$

where f is the offset and $\tan\theta(z)$ can be written in terms of the ray parameter, $p = \sin\theta(z) / v(z)$. After describing the time coordinates, we will derive an expression for $f/2$ as a function of ray parameter, rms velocity, and two-way slant frame travel time to the reflector at z_0 .

Various time coordinates are shown in Figure 3. Definitions for these coordinates are given below.

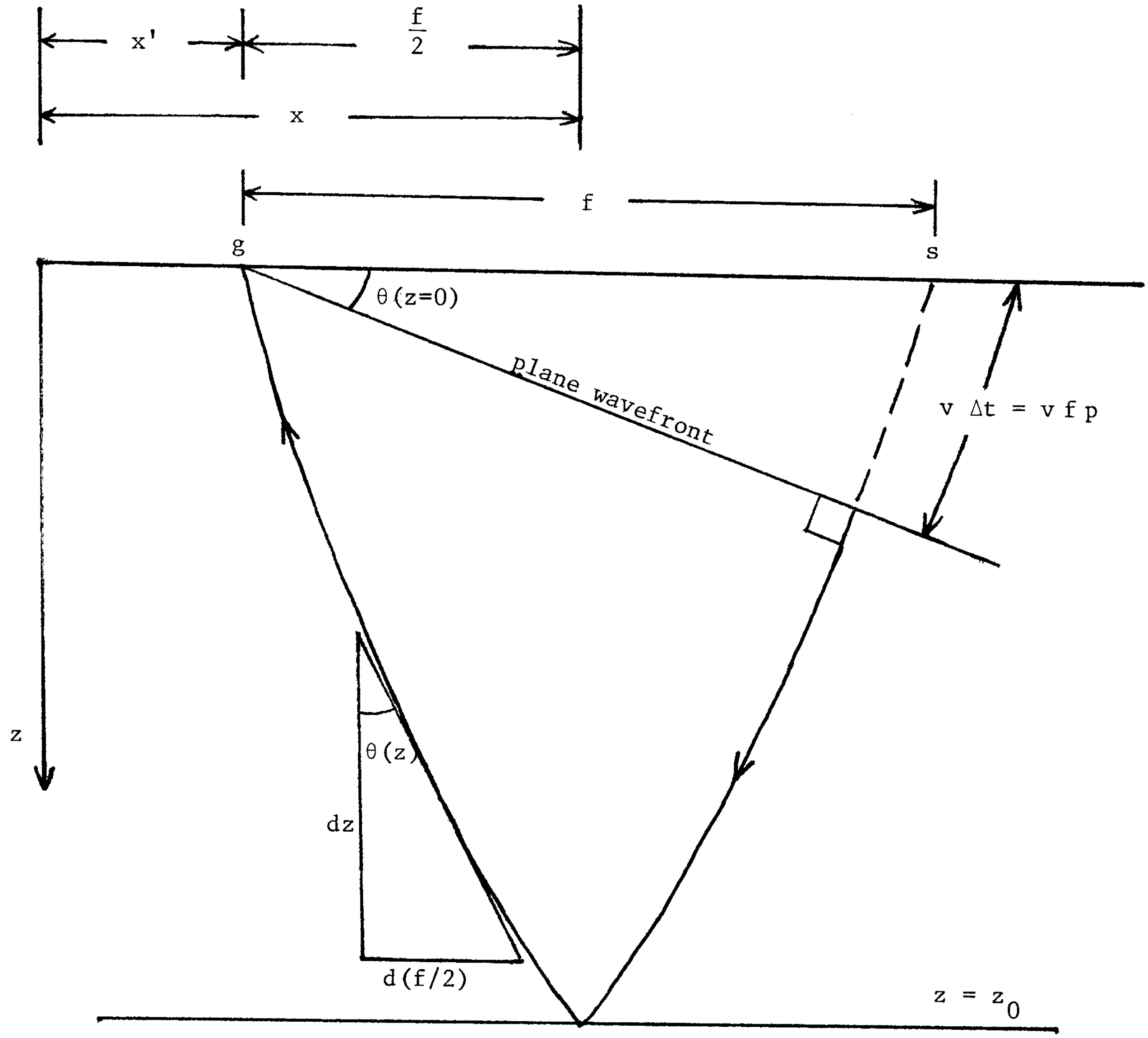

Figure 2. A ray path diagram for data which has been slant plane wave stacked in common geophone gathers. The plane wavefront source is in the $t' = 0$ position (t' defined in text and Figure 3). The surface position labeled "s" is that of the particular shot point from which most of the energy is received at the surface receiver position, "g", for this particular reflector depth and stacking parameter, p . The energy arriving at g from the ray path shown is all placed at horizontal coordinate x' by the stacking process. The dashed region of the ray path is a physical interpretation of the difference in travel times, Δt , defined in Figure 3. The triangle to the lower left shows geometrically the relation $df = 2 \tan\theta(z)\, dz$.

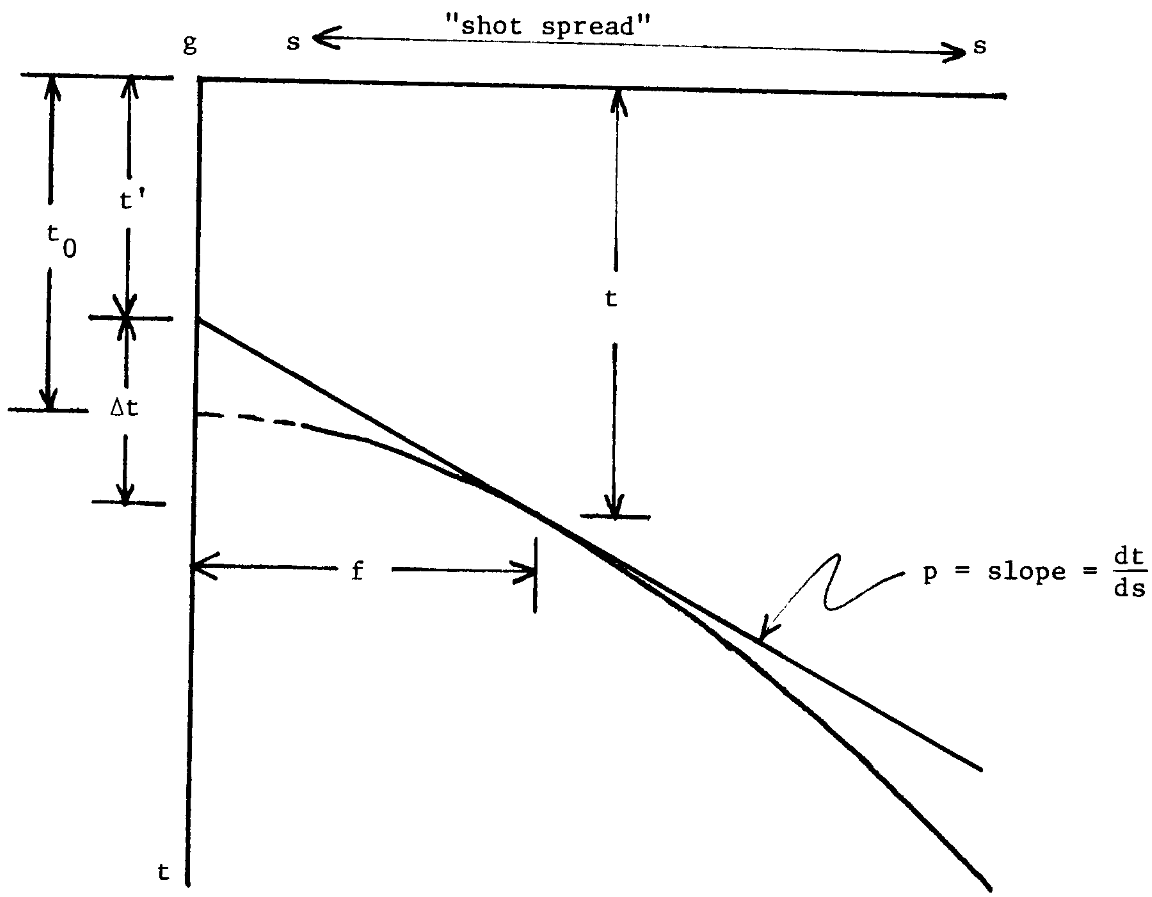

Figure 3. A single hyperbolic event shown in a common geophone gather. The line labeled "p" shows the stacking trajectory, and its intercept with the time axis defines t' for this particular event and stacking parameter p. The intervals marked f and Δt can also be seen on Figure 2 in real physical space. Figure 3 shows clearly the relationship $\Delta t = p f$.

t : the shot-to-receiver two-way travel time (ray path shown in Figure 2).

t' : the two-way travel time after slant plane wave stack with parameter p .

Δt : $t - t'$.

t_0 : the two-way travel time for a vertically traveling plane wave or a zero-offset ray path.

Following the derivation in "Velocity Estimation in Slant Frames I" (this report), we observe its equations (23) and (24), and write

$$\frac{f}{2} = \frac{p\, v_{rms}^2\, t}{2} \tag{2}$$

We write below two equations representing the most important result in "Velocity Estimation ...". That is, given the straight line ray path assumption, the travel time curves for a reflector at z_0 are

$$\frac{t'^2}{t_0^2} + p^2\, v_{rms}^2 = 1 \tag{3}$$

and

$$\frac{t_0^2}{t^2} + p^2\, v_{rms}^2 = 1 \tag{4}$$

Now, incorporating equations (3) and (4) into equation (2), we have

$$\frac{f}{2} = \frac{p\, v_{rms}^2\, t'}{2(1 - p^2\, v_{rms}^2)} \tag{5}$$

or

$$x = x' + \frac{p\, v_{rms}^2\, t'}{2(1 - p^2 v_{rms}^2)} \tag{6}$$

Equation (6) tells us that given slant stacked data of a particular value of ray parameter, p, we can transform the data from the (x', t') coordinates into the more desirable (x, t') coordinates by a t'-dependent horizontal shifting of the data. In the case of constant velocity, we can see that equation (6) predicts the uniform shearing of the data that we intuitively expected to find in view of Figure 1.

Let us generalize these results:

We have field data in (s, g, t) coordinates. After the slant stack over common geophone gathers or common shot gathers, we have the coordinates (x', p, t'), with the sign of p denoting over which type of gather the slant stack was done.

The horizontal coordinate, x', is constant along a ray, but clearly is not constant for subsurface reflectors which have the same earth-based horizontal coordinate, x. Our results here allow us to transform slant stacked data from (x', p, t') into (x, p, t') by equation (6).