Some Numerical Aspects of Stacking

by Bjorn Engquist

We will here analyze different ways of producing common shot slant stacks. The results can also be applied to other types of stacking.

Mathematically this means that we will study approximations of line integrals of a function which is known through measurements on a regular two-dimensional mesh. These measurements are themselves partial integrals of the function.

In particular, we will concentrate on the following problems. Data is recorded on a finite space interval $a \leq x \leq b$. In order to produce a slanted plane wave we would like to integrate over an infinite interval. The other problem is the approximation within the interval (a, b) when the sampling is so coarse that a simple summing procedure is not enough.

To be able to treat these problems mathematically, we will work with simple models both in the analysis and in the experiments. We believe that the results indicate solutions also for practical cases.

Our numerical problem is then to approximate

$$\int_{-\infty}^{\infty} P(px + t', x)\, dx \tag{1}$$

when we know the values

$$P_{j,k} = \frac{1}{2\varepsilon} \int_{x_k-\varepsilon}^{x_k+\varepsilon} P(t_j, x)\, dx \tag{2}$$

We say that a method is n'th order accurate if the error in the approximation is of order $O(\Delta t^n + \Delta x^n)$. We need $(n+1)$'st order accuracy when estimating the integral over the short intervals to get an n'th order method for (a, b) .

Let us approximate the integral

$$S_k = \int_{x_k}^{x_{k+1}} P(px + t', x)\, dx \tag{4}$$

by the formula

$$S_k(\Delta) = \Delta x \sum_{j=L(k)}^{U(k)} (a_{j,k} P_{j,k} + a_{j,k+1} P_{j,k+1}) \tag{5}$$

The full integral is then $S = \sum_{k=0}^{K-1} S_k$, for $x_0 = a$, $x_K = b$.

We can now use Taylor expansions to determine conditions on $a_{j,k}$ which are necessary for a second order scheme ($\varepsilon = O(\Delta x)$) .

$$P_{j,k} = \frac{1}{2\varepsilon} \int_{x_k-\varepsilon}^{x_k+\varepsilon} P(t_j, x)\, dx =$$

$$= \frac{1}{2\varepsilon} \int_{x_k-\varepsilon}^{x_k+\varepsilon} \left(P(t_j, x_k) + \frac{(x-x_k)^2}{2} P_{xx}(t_j, x_k) + O(\Delta x^4) \right) dx$$

$$= P(t_j, x_k) + \frac{\varepsilon^2}{6} P_{xx}(t_j, x_k) + O(\Delta x^4) \tag{6}$$

Since $\varepsilon = O(\Delta x)$, and in most cases $\varepsilon = \frac{\Delta x}{2}$ we have

$$P_{j,k} = P(t_j, x_k) + O(\Delta x^2)$$

When only second order methods are considered the $O(\Delta x^2)$ term can be neglected. The argument of P below is $(p\, x_{k+1/2} + t',\ x_{k+1/2})$.

$$S_k = \int_{x_k}^{x_{k+1}} (P + (px - (p\, x_{k+1/2} + t')) P_t + (x - x_{k+1/2}) P_x +$$

$$+ O(\Delta x^2))\, dx = \Delta x\, P + O(\Delta x^3) \qquad (7)$$

$$S_k(\Delta) = \Delta x \sum_j ((a_{j,k} + a_{j,k+1}) P +$$

$$+ \frac{\Delta x}{2} (- a_{j,k} + a_{j,k+1}) P_x + \Delta t\, (a_{j,k} t_{j,k} + a_{j,k+1}\, t_{j,k+1}) P_t +$$

$$+ O\,(\Delta x^2 + \Delta t^2)) \qquad (8)$$

$$t_{j,k} = (t_j - (p\, x_k + t')) / \Delta t$$

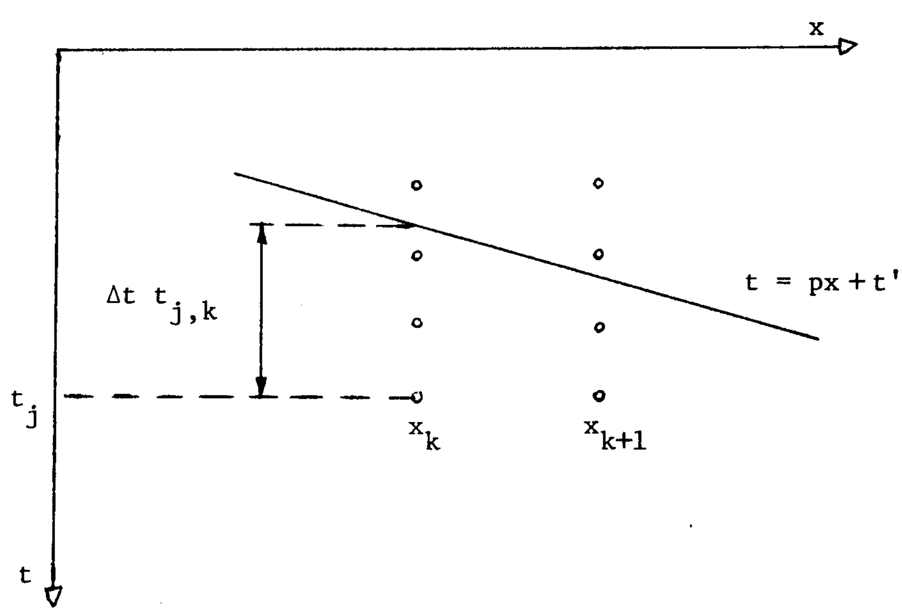

Figure 1.

In order to approximate S_k modulo $O(\Delta x^3 + \Delta x \Delta t^2)$ with $S_k(\Delta)$ we must have

$$\sum_j (a_{j,k} + a_{j,k+1}) = 1$$

$$\sum_j (a_{j,k} - a_{j,k+1}) = 0 \qquad (9)$$

$$\sum_j (a_{j,k} t_{j,k} + a_{j,k+1} t_{j,k+1}) = 0$$

We will get the simplest formula if the sum involves only two points in the t direction for each x_k. Let us look at an example

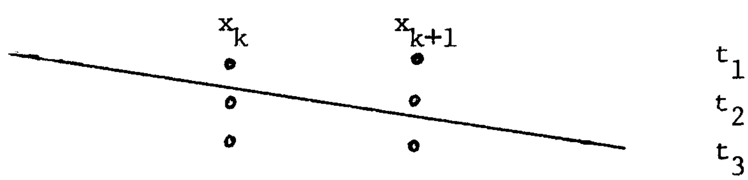

Figure 2.

$$a_{1,k} = -\frac{t_{2,k}}{2} \qquad a_{2,k} = \frac{t_{1,k}}{2}$$

$$a_{2,k+1} = -\frac{t_{3,k+1}}{2} \qquad a_{3,k+1} = \frac{t_{2,k+1}}{2}$$

This is just the trapezoidal rule with linear interpolation in the t direction. If we use one point in t, the nearest neighbor, we will have a final error of the size $O(\Delta t + \Delta x^2)$. Later, we will see how several points in t can improve the accuracy and reduce the aliasing error even if the scheme still is of second order.

If we want a higher order method the fact that $P_{j,k} \neq P(t_j, x_k)$ must be taken into account. Our local formulas will now look like

$$S_{2k} = \int_{x_{k-1}}^{x_{k+1}} P(px + t', x)\, dx$$

$$S_{2k}(\Delta) = 2\Delta x \sum_j (a_{j,k-1} P_{j,k-1} + b_{j,k} P_{j,k} + a_{j,k+1} P_{j,k+1}) \quad (10)$$

In the same way as for the second order scheme we can Taylor expand both sides and choose $a_{j,k}$ and $b_{j,k}$ in order to annihilate the lower order terms. The conditions for fourth order accuracy are the following: (For convenience we omit the Σ sign and write $a_{j,k-1} = a$, $b_{j,k} = b$, $a_{j,k+1} = c$. That is, in the first equation $a+b+c$ stands for $\sum_j (a_{j,k-1} + b_{j,k} + a_{j,k+1})$.)

$$
\begin{array}{llll}
a + b + c & = 1 & (\text{P-factor}) & \\
-a + c & = 0 & (P_x) & \\
a\,t_{j,k-1} + b\,t_{j,k} + c\,t_{j,k+1} & = 0 & (P_t) & \\
a + c + \frac{1}{3}\left(\frac{\varepsilon}{\Delta x}\right)^2 (a+b+c) & = \frac{1}{3} & (P_{xx}) & \\
-a\,t_{j,k-1} + c\,t_{j,k+1} & = \frac{2p}{3} & (P_{xt}) & (11) \\
a\,t_{j,k-1}^2 + b\,t_{j,k}^2 + c\,t_{j,k+1}^2 & = \frac{p^2}{3} & (P_{tt}) & \\
(-a + c & = 0) & (P_{xxx}) & \\
a\,t_{j,k-1} + c\,t_{j,k+1} & = 0 & (P_{xxt}) & \\
-a\,t_{j,k-1}^2 + c\,t_{j,k+1}^2 & = 0 & (P_{xtt}) & \\
a\,t_{j,k-1}^3 + b\,t_{j,k}^3 + c\,t_{j,k+1}^3 & = 0 & (P_{ttt}) &
\end{array}
$$

There are many solutions to this messy system. We will here only note that it is possible to show

$$\sum_j a_{j,k-1} = \sum_j a_{j,k+1} = \frac{1}{6} - \frac{1}{6}\left(\frac{\varepsilon}{\Delta x}\right)^2$$
$$\sum_j b_{j,k} = \frac{2}{3} + \frac{1}{3}\left(\frac{\varepsilon}{\Delta x}\right)^2 \tag{12}$$

This is a modification of the classical Simpson rule in the x direction.

We have so far used closed formulas. That is the mesh points (x_k) were also end points of the small intervals of integration (x_k, x_{k+1}) and (x_{k-1}, x_{k+1}). In the second order case there are simple types of open formulas

$$\int_{x_k-\Delta x/2}^{x_k+\Delta x/2} P(px+t', x)\,dx \sim \Delta x \sum_j a_{j,k} P_{j,k} \tag{13}$$

The conditions on $a_{j,k}$ are

$$\sum a_{j,k} = 1 \quad , \quad \sum a_{j,k} t_{j,k} = 0 \tag{14}$$

A typical case with $a_{j,k} \neq 0$ at two points in the t direction can look like

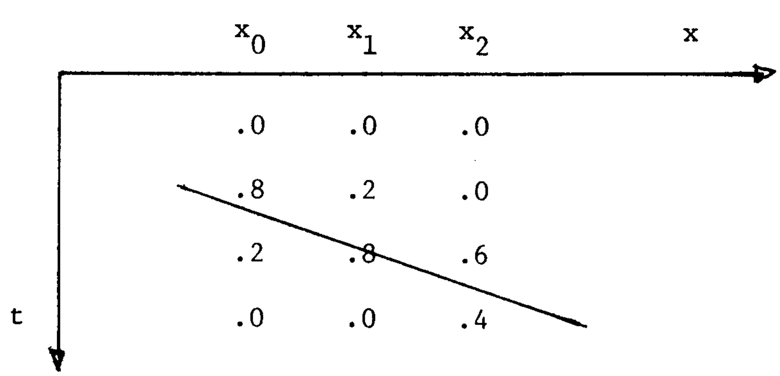

Figure 3. The values of $a_{j,k}$

The methods we have derived are valid for any P. We will now turn to the case where we know something about the structure of P and assume that locally

$$P(t,x) = f(t - p'x) + g(t,x) \tag{15}$$

where g and its derivatives are small. This means that, in a small area, P does not change so much along a line $t = p'x + t_0$. The line can be thought of as part of a hyperbola in the time versus receiver coordinate plane. An estimate of the velocity is needed to get an approximation of p'.

We will use formulas that obey conditions for second order accuracy. Even if our guess regarding the structure of P is wrong, we will still have a second order method. Our goal is to reduce the constant in front of Δx^2 in the error $O(\Delta x^2 + \Delta t^2)$. We want it to depend on the derivatives of g instead of those of $(f + g)$. In practice we see the effect of large derivatives as an aliasing error. Figure 4 shows how it may occur. See also the related paper by Philip Schultz in this volume.

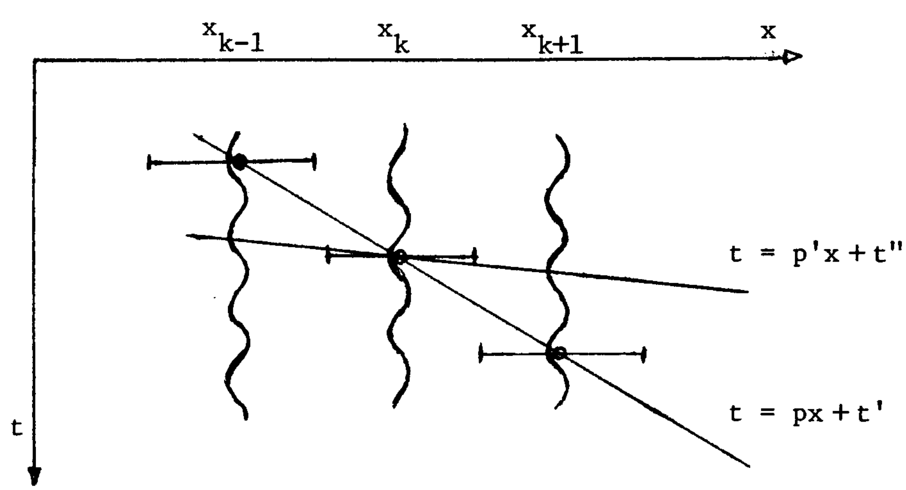

Figure 4.

In Figure 4, P_{jk} corresponds to the integral of P over ⟝●⟞ . P is described by the traces. The risk for aliasing is especially large at the inner traces. When $p > p'$ the integrals over $(x_k - \varepsilon, x_k + \varepsilon)$ does not help so much.

The sampling in x is in general, much coarser than that in t. Our approach is to improve the approximation by summing over several points in t for each x_k.

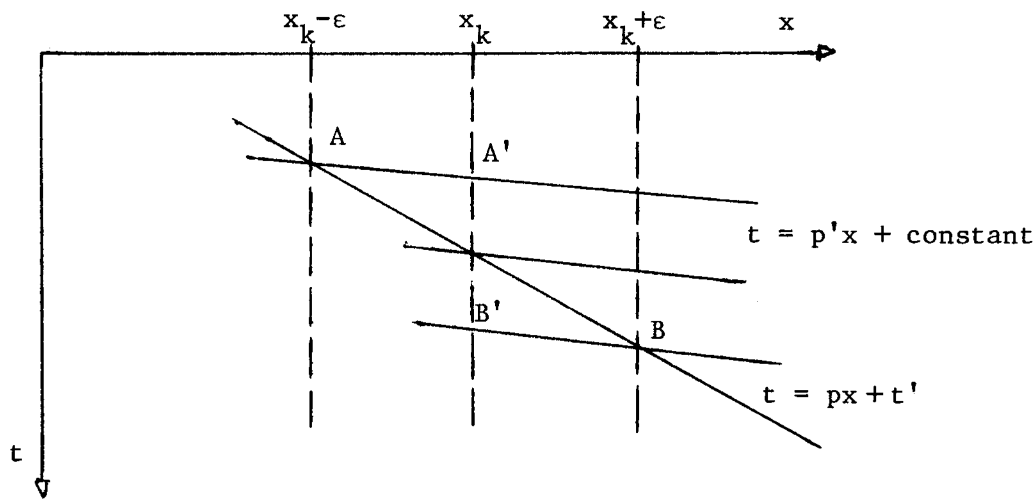

Figure 5.

In Figure 5 we can see the idea. We want the integral of P over (A,B). The function P is almost constant along lines with slope p'. Hence, P takes approximately the same values on (A', B') as on (A,B) and we can integrate over (A', B') instead.

The relation

$$\int_{(A,B)} P\,dx = \frac{1}{p-p'} \int_{(A',B')} P\,dt$$

follows from the geometry. If the sampling is dense in t and the estimate of p' is good, we can integrate accurately along (A', B') and the error in the stacking will mainly come from the g part of P .

Let us consider this procedure from a more mathematical point of view. We will work with the midpoint formula (13) , (14) , where we approximate

$$S'_k = \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} P\,(\,px + t'\,,\,x\,)\,dx$$

and first assume that $\varepsilon = 0$ and $\Delta t << \Delta x$. The integral over (a , b) is as before derived by summing over all S'_k , ($k = 0 ,\ldots, K$). In figure 3 we suggested $a_{j,k} \neq 0$ for the points closest to the line of integration. The local error in this formula is determined by using Taylor expansions.

$$S'_k(\Delta) - S'_k = \Delta x\,(\,a_{j,k}P(t_j,x_k) + a_{j+1,k}P(t_{j+1},x_k)\,) -$$

$$- \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} P(\,px + t',x\,)\,dx = \Delta x\,P(\,px_k + t',x_k\,) + O(\Delta x\Delta t^2) -$$

$$- \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} (\,P(\,px_k + t',x_k\,) + \frac{(\,x - x_k\,)^2}{2}\,\frac{\partial^2}{\partial x^2}\,(\,P(\,px_k + t',x_k))$$

$$+ O(\Delta x^4))\,dx = \frac{\Delta x^3}{24}\,(\,p - p'\,)^2\,f''(\,x_k(\,p - p'\,) + t'\,) +$$

$$+ O(\,\Delta x^3|\,g\,|_2 + \Delta x^4 + \Delta x\Delta t^2\,) \qquad (17)$$

Here $|\,g\,|_2$ denotes the sum of the maximum norms of the second derivatives of g .

We will now use several $a_{j,k} \neq 0$ in (A',B') and choose the weights according to some standard quadrature formula to approximate the integral

$$\frac{1}{p - p'} \int_{t(1)}^{t(2)} P(t,x_k) \, dt$$

$$t(1) = px_k + t' - (p - \bar{p}) \Delta x/2$$

$$t(2) = px_k + t' + (p - \bar{p}) \Delta x/2$$

$\bar{p}$ is the estimated p'

The principal part of the local error comes from the difference between p' and the guessed value $\bar{p}$. Analogous to earlier derivations we can determine the error : ($\delta = (p' - \bar{p}) \Delta x/2$)

$$\frac{1}{p - \bar{p}} \left(\int_{t(1)-\delta}^{t(1)} + \int_{t(2)}^{t(2)+\delta} \right) \frac{(t- (px_k+t'))^2}{2} f''(t-p'x_k) \, dt$$

$$+ O(\Delta x^3 | g |_2 + \Delta x^4 + \Delta x \Delta t^2) =$$

$$= \frac{\Delta x^3}{24} \frac{p'-\bar{p}}{p-\bar{p}} (3(p-\bar{p})^2 + 3(p-\bar{p})(p'-\bar{p}) + (p'-\bar{p})^2)$$

$$f''((p-p')x_k + t') + O(\Delta x^3 | g |_2 + \Delta x^4 + \Delta x \Delta t^2) \qquad (18)$$

This formula tells us how the severe part of the error depends on our estimate $\bar{p}$ of p' .

If we also want to enclude the fact that $\varepsilon \neq 0$ and that $P_{j,k}$ are integrals over ($x_k-\varepsilon$, $x_k+\varepsilon$) , we can project these intervals to $(t_j - p'\varepsilon , t_j + p'\varepsilon)$. In this way the f part of P can be treated correctly. We are left with the problem of estimating a time integral where the known function values are themselves partial time integrals. This is like our original problem and we can i.e. use formulas analogous to (12) .

Formula (17) tells us how the error grows with $p - p'$ when applying a standard method. The function P becomes more oscillatory with increasing $|p - p'|$ and the contribution to the final integral decays. This argument suggests the use of a weighting function in the integral which annihilates P for large $|p - p'|$. It is a simple and fast method, but the loss of information when tapering down the function values is a drawback. See the paper by Philip Schultz and our own discussion of tail estimations.

Estimation of the tails

There are cases where the tails contribute substantially to the integral. In practice this occurs e.g. when the point of tangency between the line of integration and a hyperbolic event is close to the end points. When the point of tangency is well in the interior of (a , b), but the line of integration passes through an event at the end point, we get another end effect. Since we do not integrate over a full wave form we get an unwanted contribution to our sum depending on at which phase we stop the integration.

There are different ways to tackle end effects. We will briefly look at the possibilities of extrapolating P along the line of integration or tapering the P values down to zero at the end points.

However, let us first consider the technique of projecting the P values off the interval (a, b) to the end point traces, where we have our measurements. In this way we will sum over several points along the first and the last traces. It is analogous to the method for the interior where we summed several $P_{j,k}$ for each fix k.

We will explain the algorithm using figure 6

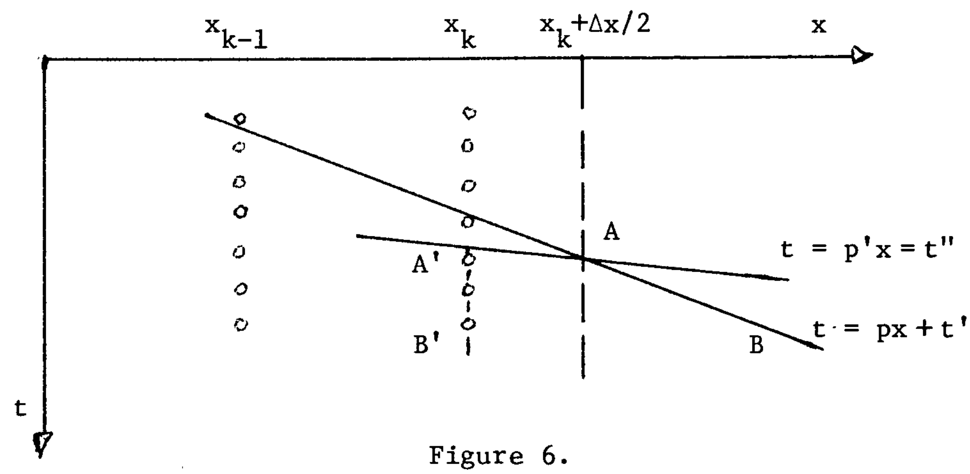

Figure 6.

Instead of summing along (A,B) we assume that $P(t,x) \sim f(t-p'x)$ and sum over (A',B') where f takes the same values. Note that if we use the midpoint formula in the interior, we shall regard $x_k + x/2$ as the end point. The relation between the integrals is

$$\int_{(A,\infty)} P\,dx = \frac{1}{p-p'} \int_{(A',\infty)} P\,dt$$

The integral $\int_{A'}^{\infty} P\,dt$ is then replaced by $\int_{A'}^{B'} r(t)P(t,x_k)\,dt$ where the weighting function r decays to zero at B'. How far B' can be taken depends on how far the approximation $P(t,x) \sim f(t-p'x)$ is valid. We have already discussed how to solve the resulting time integral over (A',B').

We can gain some insight if we apply these ideas to a model problem. We are only interested in the behavior close to the endpoint.

Define

$$S = \int_0^\infty P(0,x)\,dx \qquad \left(= \frac{c_1}{c_1^2 + (\omega p')^2} \right)$$

$$\text{where} \quad P(t,x) = e^{-c_1 t - c_2 x} \sin \omega\,(t + p'x)$$

$$\omega \gg 1$$

The primitive estimator of S is

$$E_1 = \int_0^1 P(0,x)\,dx$$

and the new one is

$$E_2 = E_1 + \frac{1}{\bar{p}} \int_0^\infty e^{-ct} P(t,1)\,dt$$

The integration can be carried out analytically and the result looks typically like

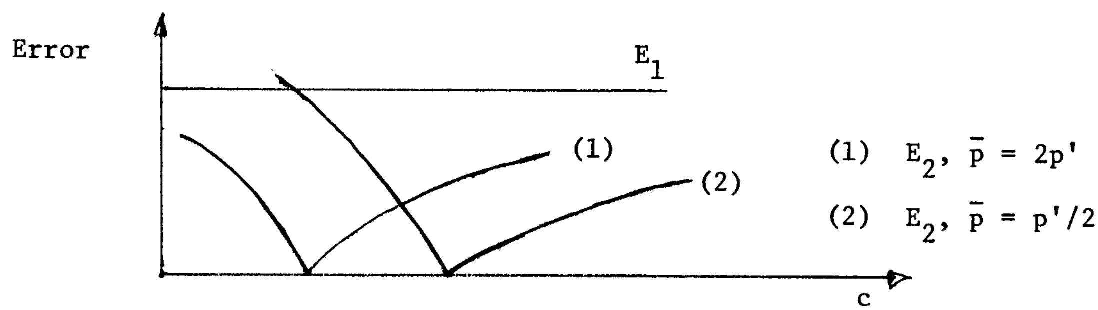

Figure 7.

We will always get some improvement when c is large enough. If we underestimate the slope p' (in general $p - p'$) we need a stronger damping than if we overestimate it. These results are valid for a wider range of problems.

We can also try to improve the algorithm . Instead of estimating the tail by the last trace we can average over a couple of traces to make the procedure more robust. The assumption $P \sim f(t - p'x)$ can be generalized to $P \sim f(t - p(x))$. There is also the possibility of using adaptive methods i.e. to let the program determine p' and the change of P along a line with slope p' .

Let us now turn to the extrapolation approach and write $P(t,x)$ along the line of integration as a function $P(s)$. One way to estimate the integral of P over (b,∞) is to extrapolate P into the unknown region. That is, using the values of P for $s < b$ to estimate the behavior of the function in some interval $b \leq x \leq b'$. We can e.g. approximate the derivatives $P_s(b)$ and $P_{ss}(b)$ and use the estimator

$$\int_b^\infty P\,ds \sim \int_b^{b'} ((P(b) + sP_s(b) + \frac{s^2}{2}P_{ss}(b)) r(s)\,ds$$

where the weighting function $r(s)$ equals 1 at b and 0 at b' . This technique has been used in other contexts, but has the disadvantage that extrapolation is very sensitive to perturbations.

Finally we have the possibility of using a weighting function in the interior to smoothly take the value of P to zero at the end points a,b . This is not an estimate of the tail integrals, but can help if P is highly oscillatory along the line of integration.

Let us look at a simple exampel in figure 8. It shows the dilemma that we are in. The weighting function must affect several wave lengthes to be efficient. The risk is then of course that the tapering reach the essential parts of the integral. We also see that when the end effect is reduced in size it affects a larger area.

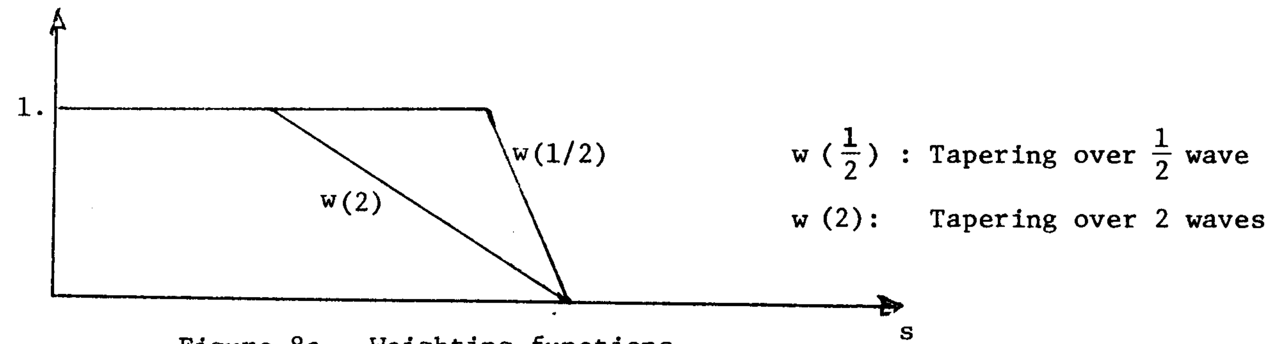

Figure 8a. Weighting functions.

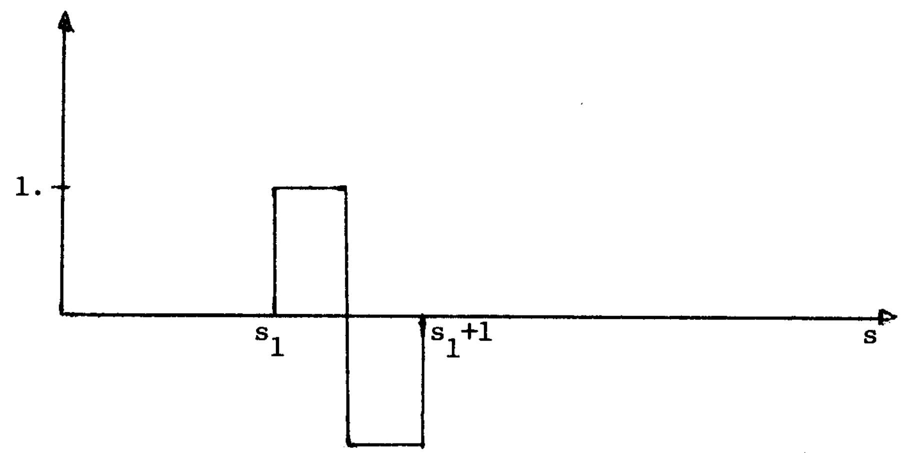

Figure 8b. The wave form.

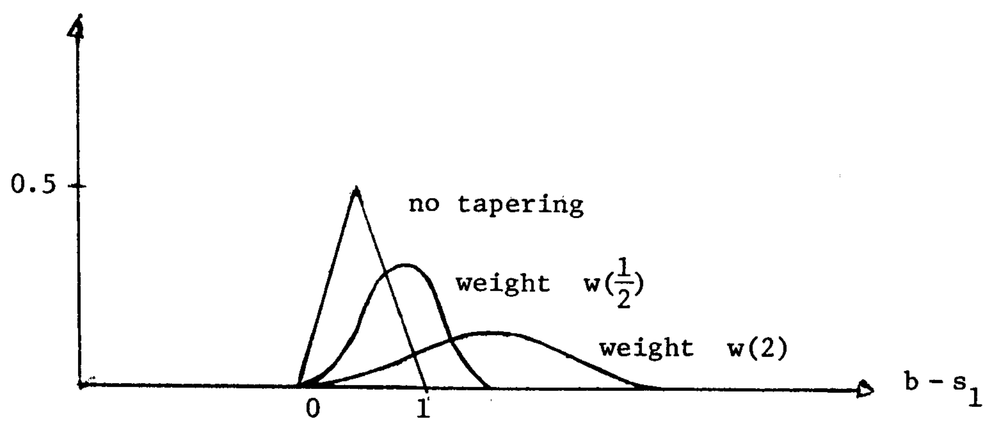

Figure 8c. Value of the weighted integral as a function of the right end point related to the wave. The desired value is zero.

Our main interests in this paper are theoretical analysis and speculations, but we have also run a few simple experiments. We started with a synthetic common shot gather with one hyperbolic event. The wave form was 1.5 periods of a sine function with zero mean value. We used 12 points per wave in time and a minimum of 5 points per wave in the offset x. The $P_{j,k}$ was produced by summing on a finer grid.

When the line of integration was tangent to the event the improvement from the nearest neighbour method to linear interpolation in t was only about 5%. At the inner traces the ailiasing error was reduced 5 - 10%. In the latter case the proposed method of summing over 5 points for each trace reduced the error 40 - 85%. Summing over 15 points in the first trace reduced the end effect 50 - 95%. The estimate of the slope p' was then off by 10%.