# Well-Posedness of One Way Wave Equations

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From the scalar wave equation

$$\frac{1}{v^2} P_{tt} = P_{xx} + P_{zz}$$

with the corresponding dispersion relation

$$\frac{1}{v^2} \omega^2 = k_x^{\,2} + k_z^{\,2}$$

we get the dispersion relation for the one way wave equation

$$k_z = +\left( \frac{\omega^2}{v^2} - k_x^{\,2} \right)^{1/2} \qquad (1)$$

In a moving coordinate frame ( $t' = t - \frac{z}{v}$ , $k_z = k_z' + \frac{\omega'}{v}$ ; $-\omega'$ is the dual of $t'$ ) we have the following relation if the primes are dropped

$$k_z = \left( \frac{\omega^2}{v^2} - k_x^{\,2} \right)^{1/2} - \frac{\omega}{v} \qquad (2)$$

We want partial differential equations (PDE's) which have dispersion relations approximating (1) or (2). We also want the PDE's to be well posed as initial value problems both in $z$ and $t$ .

What do we mean by well-posedness, eg., when $z$ is the evolution direction? In general, we want a reasonable norm of the solution to be bounded by a constant times the norms of the initial values. The constant may not depend on $\omega$ or $k_x$ . What we more explicitly mean will become clear in our examples. We will also consider a weak form of well-posedness: After Fourier transforming in $t$ and $x$ , the solution is not allowed to grow exponentially in $\omega$ and $k_x$ . If the solutions are of the form $\sim \exp(c(\omega,k_x)z)$ then the real part of $c$ , ( Re(c) ), shall have an upper bound. We have an ill posed problem if Re(c) can be arbitrarily large. The situation is analogous with $t$ as evolution direction.

The square root in (1) and (2) can be approximated by a polynomial or a rational function. We will see that a three term Taylor expansion of the square root generates PDE's which are not well posed as initial value problems with $t$ as evolution direction. In a paper to be published, Francis Muir uses continued fraction approximations to produce approximating PDE's of any order. We will here show that his approach generates equations which do not suffer from the same weakness. Finally, we will consider one way wave equations in slanted frames.

A natural beginning when approximating the square root is to use a Taylor expansion

$$(1-X^2)^{1/2} = 1 - \frac{X^2}{2} - \frac{X^4}{8} - \frac{X^6}{16} - \cdots$$

When this is applied to (1) and (2) where we expand around $\frac{v\,k_x}{\omega} = 0$ we get respectively (the second equation corresponds to primed coordinates),

$$k_z = \frac{\omega}{v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{1/2} = \frac{\omega}{v} - \frac{v\,k_x^2}{2\,\omega} - \frac{v^3 k_x^4}{8\,\omega^3} - \cdots$$

$$k_z = \frac{\omega}{v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{1/2} - \frac{\omega}{v} = - \frac{v\,k_x^2}{2\,\omega} - \frac{v^3 k_x}{8\,\omega^3} - \cdots$$

The corresponding PDE's for second and third order approximation are (equation (5) and (6) are in the primed coordinates),

$$P_{tz} + \frac{1}{v} P_{tt} - \frac{v}{2} P_{xx} = 0 \tag{3}$$

$$P_{tttz} + \frac{1}{v} P_{tttt} - \frac{v}{2} P_{ttxx} - \frac{v^3}{8} P_{xxxx} = 0 \tag{4}$$

$$P_{tz} - \frac{v}{2} P_{xx} = 0 \tag{5}$$

$$P_{tttz} - \frac{v}{2} P_{ttxx} - \frac{v^3}{8} P_{xxxx} = 0 \tag{6}$$

It is easy to see that all these equations are well posed if $z$ is the evolution direction. After Fourier transforming in $x$ and $t$ they are of the form

$$\hat{P}_z(\omega, k_x, z) = -i\, a(\omega, k_x)\, \hat{P}(\omega, k_x, z)$$

where $a(\omega, k_x)$ is real. This implies

$$|\hat{P}(\omega, k_x, z)| = |\hat{P}(\omega, k_x, 0)|$$

and from Parseval's relation we get the energy conservation,

$$\| P(\ ,\ , z) \| = \| P(\ ,\ , 0) \| \tag{7}$$

( Here $\| \ \|$ denotes the $L_2$ norm . )

Now let $t$ be the evolution direction and note that, if the PDE is of order $n$ in time, we need $n$ initial conditions. The values of $P$ , $\frac{\partial}{\partial t} P$ , ... , $\frac{\partial^{n-1}}{\partial t^{n-1}}$ at $t=0$ are assumed to be given.

We will see that equations (3) and (5), but not equations (4) and (6), are well posed as initial value problems. Fourier transforming in $x$ and $z$ gives ordinary differential equations in $t$ . In order to check the growth of their solutions we consider the corresponding characteristic equations ( $\partial_t \to s$ , $\partial_{tt} \to s^2$ , ... ). They are respectively,

$$s^2 + i v k_z s + \frac{v^2 k_x^2}{2} = 0 \qquad (3')$$

$$s^4 + i v k_z s^3 + \frac{v^2 k_x^2}{2} s^2 - \frac{v^4 k_x^4}{8} = 0 \qquad (4')$$

$$s - i \frac{v k_x^2}{2 k_z} = 0 \qquad (5')$$

$$s^3 - i \frac{v k_x^2}{2 k_z} s^2 + i \frac{v^3 k_x^4}{8 k_z} = 0 \qquad (6')$$

If a characteristic equation has the roots $s_1, \ldots, s_n$, then the solution to the corresponding ordinary differential equation can be written in the general form

$$\hat{P}(t, k_x, k_z) = \sum_{j=1}^{n} A_j \exp(s_j t)$$

when the roots are separate. The coefficients ( $A_j = A_j(k_x, k_z)$) are determined by the initial conditions. It is necessary for well posedness to bound the growth of the exponentials, i.e., to have an upper bound on the real part of $s_j$ : $Re(s_j) \leq C$ , where $C$ is independent of $k_x$ and $k_z$ .

Let us consider (3'). The characteristic equation has two imaginary roots which are separate for $k_x k_z \neq 0$ .

$$s = i v \left( \frac{k_z}{2} \pm \left( \frac{k_z^2}{4} + \frac{k_x^2}{2} \right)^{1/2} \right)$$

Hence, there is no expenential type of ill posedness. In order to get an explicit bound of the solution we write its general form in the following way

$$\hat{P}(t) = A(\exp(s_1 t) + \exp(s_2 t)) + B(\exp(s_1 t) - \exp(s_2 t))$$

where A and B are determined by the initial conditions ( at t=0 )

$$2A = \hat{P}(0)$$

$$A(s_1 + s_2) + B(s_1 - s_2) = \hat{P}_t(0)$$

$$A = \frac{\hat{P}(0)}{2}$$

$$B = \frac{\hat{P}_t(0)}{s_1 - s_2} - \frac{s_1 + s_2}{2(s_1 - s_2)} \hat{P}(0)$$

This gives us (after multiplying with $\exp(-s_2 t)$)

$$|\hat{P}(t)| \leq \left| \frac{\exp((s_1-s_2)t) + 1}{2} - \frac{(s_1+s_2)(\exp((s_1-s_2)t)-1)}{2(s_1 - s_2)} \right| |\hat{P}(0)|$$

$$+ \frac{|\exp((s_1-s_2)t) - 1|}{|s_1 - s_2|} |\hat{P}_t(0)| \leq \frac{|s_2 \exp((s_1-s_2)t) - s_1|}{|s_1 - s_2|} |\hat{P}(0)| +$$

$$+ t |\hat{P}_t(0)| \leq |\hat{P}(0)| + t |\hat{P}_t(0)|$$

and we have the estimate

$$||P(t, , )|| \leq ||P(0, , )|| + t ||P_t(0, , )||$$

Consider equation (4'): For $k_z \to 0$ the equation approaches

$$(s^2)^2 + \frac{v^2 k_x^2}{2} s^2 - \frac{v^4 k_x^4}{8} = 0$$

$$s^2 = v^2 k_x^2 \left( -\frac{1}{4} \pm \left( \frac{1}{16} + \frac{1}{8} \right)^{1/2} \right)$$

One root $s = \frac{v}{2}|k_x|\ (\sqrt{3}-1)^{1/2}$. There is no upper bound for Re(s) and (4) cannot be well posed.

Equation (5) is well posed since (5') has only one imaginary root. We get an estimate analogous to (7).

Considering equation (6'): For $k_z = k_x^3$ and $k_x \to \infty$ the equation approaches

$$s^3 + i\,\frac{v^3}{8}\,k_x = 0$$

The real part of one root $\to \infty$ when $k_x \to \infty$ and hence (6) is not well posed.

We have seen that certain Fourier modes in the solutions to (4) and (6) grow arbitrarily fast. These problems cannot be approximated with stable difference approximations.

Let us now turn to rational approximations in hope for useful higher order equations. The following approximation of the square root

$$(1-X^2)^{1/2} \approx \frac{1-\frac{3}{4}X^2}{1-\frac{1}{4}X^2} = 1 - \frac{X^2}{2} - \frac{X^4}{8} - \frac{X^6}{32} - \ldots$$

has been used earlier in SEP to produce PDE's corresponding to (1) and (2)

$$P_{ztt} - \frac{v^2}{4}P_{zxx} + \frac{1}{v}P_{ttt} - \frac{3v}{4}P_{xxt} = 0 \qquad (8)$$

$$P_{ztt} - \frac{v^2}{4}P_{zxx} - \frac{v}{2}P_{xxt} = 0 \qquad (9)$$

Let us look at the PDE in the primed coordinates (9). It has the characteristic equation

$$s^2 - i \frac{v k_x^2}{2 k_z} s + \frac{v^2 k_x^2}{4} = 0$$

with the roots

$$s = i \left( \frac{v k_x^2}{4 k_z} \pm \left( \frac{v^2 k_x^4}{16 k_z^2} + \frac{v^2 k_x^2}{4} \right)^{1/2} \right)$$

This is analogous to (3') and we get the estimate

$$\| P(t, \ , \ ) \| \leq \| P(0, \ , \ ) \| + t \, \| P_t(0, \ , \ ) \|$$

As in the earlier problems this one is energy conserving in the $z$ direction.

Equations (8) and (9) can be regarded as the third order formulas when using continued fraction approximation of the square root as suggested by Muir.

$$(1 - X^2)^{1/2} = 1 - \cfrac{X^2}{2 - \cfrac{X^2}{2 - X^2 \cdots}}$$

Denote the finite approximation $S_j = S_j(X^2)$

$$S_{j+1} = 1 - \frac{X^2}{1+S_j} \qquad \left( = 1 + \frac{T_{j+1}}{B_{j+1}} \right) \tag{10}$$

$$S_1 = 1$$

For $T$ and $B$ we have

$$\frac{T_{j+1}}{B_{j+1}} = \frac{-X^2}{1 + 1 + \frac{T_j}{B_j}} = \frac{-X^2 \, B_j}{2 B_j + T_j}$$

By using the same formulas for $T_j / B_j$ , we can derive

$$T_{j+1} = 2\,T_j - X^2\,T_{j-1}$$
$$B_{j+1} = 2\,B_j - X^2\,B_{j-1} \tag{11}$$

Let us apply these approximations to the one way wave dispersion relation in moving coordinate frame (2). From

$$k_z = \frac{\omega}{v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{1/2} - \frac{\omega}{v}$$

we get

$$k_z - \frac{\omega\,T_j}{v\,B_j} = 0$$

$$\left( T_j = T_j(-X^2),\ B_j = B_j(-X^2),\ X^2 = \frac{v^2 k_x^2}{\omega^2} \right)$$

As our dispersion relation we will use

$$D_j = \left(-\frac{\omega}{2}\right)^j \left( k_z\,B_j - \frac{\omega}{v}\,T_j \right) = 0$$

We can now use the relation (11)

$$D_{j+1} = \left(-i\frac{\omega}{2}\right)^{j+1} \left( k_z\,B_{j+1} - \frac{\omega}{v}\,T_{j+1} \right) =$$

$$= -i\omega\left( \left(-i\,\frac{\omega}{2}\right)^j \left( k_z\,B_j - \frac{\omega}{v}\,B_j \right)\right) +$$

$$+ \frac{X^2\omega^2}{4}\left( \left(-i\frac{\omega}{2}\right)^{j-1} \left( k_z\,B_{j-1} - \frac{\omega}{v}\,B_{j-1} \right)\right) =$$

$$= -i\omega\,D_j + \frac{v^2 k_x^2}{4}\,D_{j-1}$$

The corresponding recurrence relation for the PDE's ( $D_j(P) = 0$ ) given by Muir will hence be

$$D_{j+1}(P) = \frac{\partial}{\partial t} D_j(P) - \frac{v^2}{4} \frac{\partial^2}{\partial x^2} D_{j-1}(P) \tag{12}$$

$$\begin{aligned} D_1(P) &= P_z \\ D_2(P) &= P_{zt} - \frac{v}{2} P_{xx} \\ ( D_3(P) &= P_{ztt} - \frac{v}{2} P_{txx} - \frac{v^2}{4} P_{zxx} ) \end{aligned} \tag{13}$$

If we do not use the primed coordinates, but stay in the original coordinate frame, we will have the same recursion relation (12) but other starting values.

$$\begin{aligned} D_1(P) &= P_t + v P_z \\ D_2(P) &= P_{tt} + v P_{tz} - \frac{v^2}{2} P_{xx} \end{aligned} \tag{14}$$

We will now show that the equations generated by (12), (13) and (12), (14) are not exponentially ill posed like (4) and (6). As before, we Fourier transform in x and t and want the corresponding characteristic equations to have pure imaginary roots. These equations are given by ( $k_z \neq 0$ )

$$d_{j+1} = s\, d_j + \frac{v^2 k_x^2}{4} d_{j-1} \tag{12'}$$

$$\begin{aligned} d_2 &= s - i \frac{v k_x^2}{2 k_z} \\ d_3 &= s^2 - i \frac{v k_x^2}{2 k_z} s + \frac{v^2 k_x^2}{4} \end{aligned} \tag{13'}$$

$$d_1 = s + i\,v\,k_z$$
$$d_2 = s^2 + i\,v\,k_z s + \frac{v^2\,k_x^{\,2}}{2} \qquad (14')$$

We see that for $k_x, k_z \neq 0$ both (13') and (14') have the structure

$$d_1 = s - i\,a$$

$$d_2 = s^2 - i\,a\,s + b$$

where $a$ and $b$ are real, $b > 0$ . ( In (13') we actually started with $j = 2$ and $3$ .) To simplify the proof we make the change of variables

$$d_j = (-i)^j\,e_j \quad , \quad s = i\,r$$

and get a recursion formula for the polynomials $e_j(r)$ .

$$e_{j+1} = r\,e_j - c\,e_{j-1} \qquad (15)$$

$$e_1 = r - a$$
$$e_2 = r^2 - ar - b \qquad (16)$$

Here $a$, $b$ and $c$ are real $b, c > 0$, ( $c = \frac{v^2\,k_x^{\,2}}{4}$ ). We want to show that the roots of $e_j(r) = 0$ , which we call $r_k^{(j)}$ , $k = 1, \ldots, j$, are all real.

$$r_1^{(1)} = a$$

$$r_1^{(2)} = \frac{a}{2} - \left( \frac{a^2}{4} + b \right)^{1/2}$$

$$r_2^{(2)} = \frac{a}{2} + \left( \frac{a^2}{4} + b \right)^{1/2}$$

We have $r_1^{(2)} < r_1^{(1)} < r_2^{(2)}$ and we will see that this type of relation is valid in general.

Let us assume that the conditions

$$r_1^{(j)} < r_1^{(j-1)} < r_2^{(j)} < \ldots < r_{j-1}^{(j-1)} < r_j^{(j)} \qquad (17)$$

are valid for the roots of $e_{j-1}(r) = 0$ and $e_j(r) = 0$. This implies

$$e_{j+1}(r_j^{(j)}) = r_j^{(j)} e_j(r_j^{(j)}) - c\, e_{j-1}(r_j^{(j)}) = -c\, e_{j-1}(r_j^{(j)}) < 0$$

since the coefficient in front of the leading power in any $e_j(r)$ is positive, and hence, $e_{j-1}(r) > 0$ for $r > r_{j-1}^{(j-1)}$.

Similarly, we get

$$e_{j+1}(r_{j-1}^{(j)}) > 0$$

$$e_{j+1}(r_{j-2}^{(j)}) < 0$$

$$\ldots$$

$$e_{j+1}(r_1^{(j)}) \begin{cases} > 0 & \text{if } j \text{ is even} \\ < 0 & \text{if } j \text{ is odd} \end{cases}$$

Further, if $K$ is large enough we have

$$e_{j+1}(K) > 0$$

$$e_{j+1}(-K) \begin{cases} < 0 & \text{if } j \text{ is even} \\ > 0 & \text{if } j \text{ is odd} \end{cases}$$

From this we see that $e_{j+1}$ has $j+1$ real roots and relation (17) is valid when $j \rightarrow j+1$ . By induction, all $e_j(r) = 0$ have real roots.

In the same way as before it is easy to check that the equations generated by (12), (13) and (12), (14) are energy conserving with $z$ as evolution direction.

So far we have made expansions around $\frac{v k_x}{\omega} = 0$ . We will now consider approximations of slanted waves, $\frac{v k_x}{\omega} \approx \sin\theta$ . See Claerbout, SEP-7, p. 30 for the appropriate coordinate transformation.

$$x' = x + z \tan\theta$$

$$z' = z$$

$$t' = t + \frac{z}{v} \cos\theta - \frac{x}{v} \sin\theta$$

$$k_x = k_x' + \omega' \frac{\sin\theta}{v}$$

$$k_z = k_z' + k_x' \tan\theta - \omega' \frac{\cos\theta}{v}$$

$$\omega = \omega'$$

The dispersion relation for upcoming waves

$$k_z = - \left( \frac{\omega^2}{v^2} - k_x{}^2 \right)^{1/2}$$

will in the new coordinates become

$$k_z' + k_x' \tan\theta - \omega' \frac{\cos\theta}{v} = - \left( \frac{\omega'^2}{v^2} - \left( k_x' + \omega' \frac{\sin\theta}{v} \right)^2 \right)^{1/2} =$$

$$= - \frac{\omega'}{v} \cos\theta \left( 1 - \frac{1}{\cos^2\theta} \left( \frac{v^2 k_x'^2}{\omega'^2} + \frac{2v \sin\theta \, k_x'}{\omega'} \right) \right)^{1/2}$$

We can here use the continued fraction approximation of the square root to produce rational dispersion relations; see Claerbout, SEP-8, p.20. The value of $\frac{v\,k_x}{\omega} - \sin\theta = \frac{v\,k'_x}{\omega'}$ is regarded as small.

$$k'_z + k'_x \tan\theta - \omega' \frac{\cos\theta}{v} = -\frac{\omega'}{v} \cos\theta \; S_j$$

where

$$S_{j+1} = 1 - \frac{X^2}{1+S_j} \quad , \quad X^2 = \frac{2v \tan\theta \; k'_x}{\cos\theta \; \omega'} + \frac{v^2 \; k'^2_x}{\cos^2\theta \; \omega'^2} \qquad (18)$$

From (18) we can derive a recursion formula for the corresponding PDE's in an analogous way to what we did in the case $\theta = 0$ .

$$D_{j+1}(P') = \frac{\partial}{\partial t'} D_j(P') + \left( \frac{v \tan\theta}{2 \cos\theta} \frac{\partial^2}{\partial x' \, \partial t'} - \frac{v^2}{4 \cos^2\theta} \frac{\partial^2}{\partial x'^2} \right) D_{j-1}(P')$$

There are different ways of starting the recursion (18). In SEP-8, p. 23 , Claerbout uses

$$S_1 = 1 - \frac{v \tan\theta \; k'_x}{\cos\theta \; \omega'} \qquad (19)$$

The PDE's corresponding to $S_2$ and $S_3$ will then be

$$P'_{t'z'} + \frac{v \tan\theta}{2 \cos\theta} P'_{x'z'} + \frac{v}{2 \cos^3\theta} P'_{x'x'} = 0 \qquad (20)$$

$$P'_{t't'z'} + \frac{v \tan\theta}{\cos\theta} P'_{t'x'z'} - \frac{v^2}{4\cos^2\theta} P'_{x'x'z'} + \frac{v}{2\cos^3\theta} P'_{t'x'x'} = 0 \qquad (21)$$

In earlier reports different equations in slanted coordinates have been given. Estevez derives the PDE

$$P'_{t'z'} + \frac{v \tan\theta}{\cos\theta} P'_{x'z'} + \frac{v^2}{2\cos^3\theta} P'_{x'x'} = 0 \qquad (22)$$

in SEP-5, p. 30, and in SEP-7, p. 32, Claerbout derives

$$P'_{t'z'} + \frac{v^2}{2\cos^3\theta} P'_{x'x'} = 0 \qquad (23)$$

These equations can be generated from $S_2$ when $S_1$ is respectively

$$S_1 = 1 - \frac{2 v \tan\theta \, k'_x}{\cos\theta \; \omega'} \qquad (24)$$

$$S_1 = 1 \qquad (25)$$

Formula (19) gives the closest fit to the square root.

Finally, we have the question of well posedness for the problems in slanted coordinates. All PDE's mentioned here have characteristic equations of a form we already have studied. Equations (20), (22) and (23) are first order in $t$ and $z$, and like equation (5), they are energy conserving in both these directions. Equation (21) is energy conserving in $z$. It is also strongly well posed as an initial value problem in $t$ since the characteristic equation

$$s^2 + i\left( \frac{v \tan\theta \, k'_x}{\cos\theta} + \frac{v \, k'^2_x}{2\cos^3\theta \, k'_z} \right) s + \frac{v^2 \, k'^2_x}{4\cos^2\theta} = 0$$

have the same structure as (3'). The same arguments are valid if we change $z$ to $-z$ or $t$ to $-t$.