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The "Star Product" in a Wave Propagation Model

by Benjamin Friedlander

A possible model for two-dimensional wave propagation is an inductor-capacitor network of the following form:

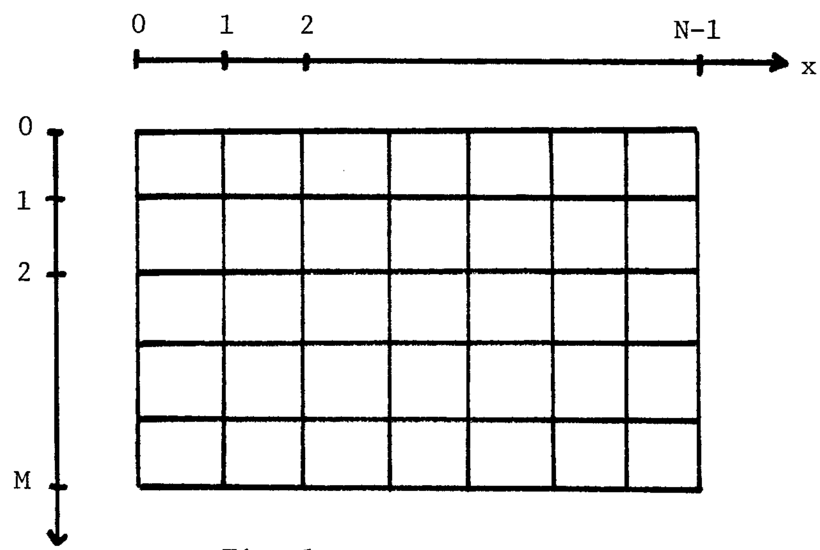

Fig. 1. The L-C Network

Each branch in this graph represents an inductor, and from each node there is a capacitor connected to some common point ("ground").

The following picture shows in detail one layer of the network.

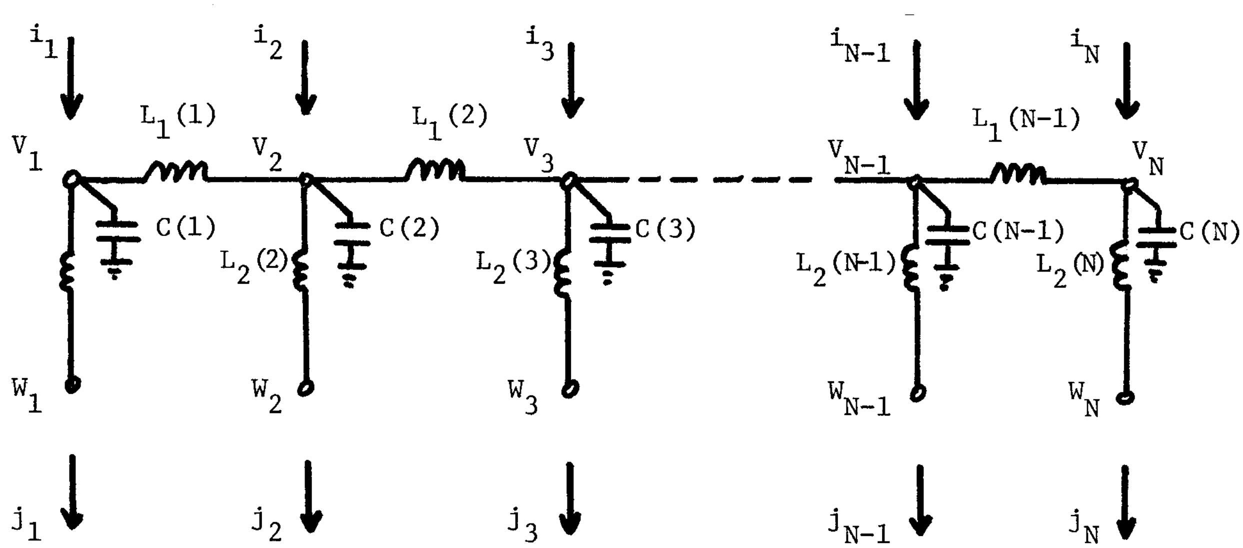

Fig. 2. A single layer. $i_1 \ldots i_N$, $V_1 \ldots V_N$ represent input currents and voltages and $j_1 \ldots j_N$, $W_1 \ldots W_N$ represent output currents and voltages.

The relations between these quantities are given by the following formulas:

Define:

$$I = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix} \quad \text{and similarly} \quad J, V, W .$$

$$I_N = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$S = i\omega = i\,2\pi f \quad \text{where } f \text{ is frequency}$$

$$R = \begin{bmatrix} \frac{1}{SL_1(1)} + SC(1) & -\frac{1}{SL(1)} & 0 & 0 \\ -\frac{1}{SL_1(1)} & \frac{1}{SL_1(1)} + \frac{1}{SL_1(2)} + SC(2) & \ddots & 0 \\ 0 & \ddots & \ddots & -\frac{1}{SL_1(N-1)} \\ 0 & 0 & -\frac{1}{SL_1(N-1)} & \frac{1}{SL_1(N-1)} + SC(N-1) \end{bmatrix}$$

$$Q = \begin{bmatrix} SL_2(1) & & 0 \\ & \ddots & \\ 0 & & SL_2(N) \end{bmatrix}$$

then

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} I_N + RQ & R \\ Q & I_N \end{bmatrix} \begin{bmatrix} J \\ W \end{bmatrix} \qquad (1)$$

or

$$\begin{bmatrix} J \\ V \end{bmatrix} = \begin{bmatrix} [I_N + RQ]^{-1} & -[I_N + RQ]^{-1} R \\ Q[I_N + RQ]^{-1} & I_N - Q[I_N + RQ]^{-1} R \end{bmatrix} \begin{bmatrix} I \\ W \end{bmatrix} \qquad (2)$$

These equations describe what happens in a single layer. What if we have many layers? The picture is as follows:

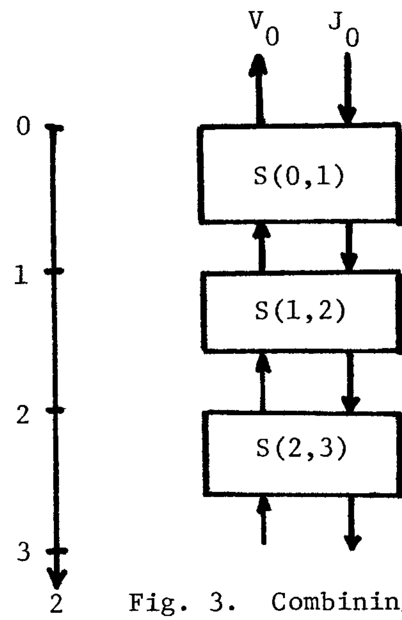

Fig. 3. Combining the layers.

and

$$\begin{bmatrix} J_{k+1} \\ V_k \end{bmatrix} = S(k,k+1) \begin{bmatrix} J_k \\ V_{k+1} \end{bmatrix} \tag{3}$$

$$S(k,k+1) = \begin{bmatrix} [I_N + R_k Q_k]^{-1} & -[I_N + R_k Q_k]^{-1} R_k \\ Q_k [I_N + R_k Q_k]^{-1} & I_N - Q_k [I_N + R_k Q_k]^{-1} R_k \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{bmatrix}$$

where R_k, Q_k are the R, Q matrices as defined before, for the k^{th} layer.

Define $S(0,k)$ by

$$\begin{bmatrix} J_k \\ V_0 \end{bmatrix} = S(0,k) \begin{bmatrix} J_0 \\ V_k \end{bmatrix} = \begin{bmatrix} S_{11}(0,k) & S_{12}(0,k) \\ S_{21}(0,k) & S_{22}(0,k) \end{bmatrix} \begin{bmatrix} J_0 \\ V_k \end{bmatrix} \tag{4}$$

and assume that the M^{th} layer is at zero potential, i.e., $V_M \equiv 0$. Then by (4) we have:

$$V_0 = S_{21}(0,M)\ J_0$$

In other words, the $S_{21}(0,M)$ matrix contains all the information we can get from making "surface measurements" (i.e., voltage/current measurements at $z=0$). There are two main problems to which we can apply this framework:

1) the forward problem: knowing the medium parameters (L_1, L_2, C) compute the surface data (synthetic data), i.e., the $S_{21}(0,M)$ matrix.
2) the backward problem: having the data, estimate the medium parameter.

The solution of the forward problem is straightforward and uses the following fact:

$$S(0,k+1) = S(0,k) * S(k,k+1) \tag{5}$$

where $*$ is called the "star product" and is defined by

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} * \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11}[I-S_{12}M_{21}]^{-1}S_{11} & M_{12}+M_{11}S_{12}[I-M_{21}S_{12}]^{-1}M_{22} \\ S_{21}+S_{22}M_{21}[I-S_{12}M_{21}]^{-1}S_{11} & S_{22}[I-M_{21}S_{12}]^{-1}M_{22} \end{bmatrix} \tag{6}$$

Equation (5) gives a recursive equation for getting $S(0,M)$ from knowledge of $S(k,k+1)$.

In terms of a geophysical problem this procedure will give the results of the following experiment

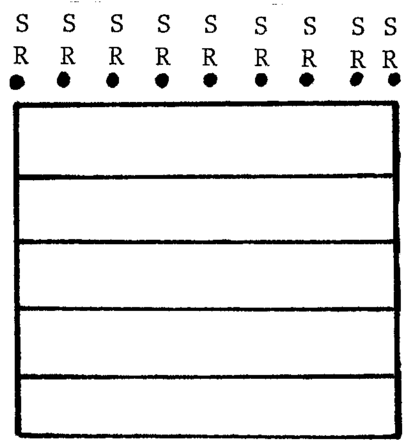

Fig. 4.

We have N shot points and N receivers as in Fig. 4. Both the velocity and the reflection coefficient of the earth are known (the medium may be nonhomogeneous both in x and z). For each of the shots what will the various receivers read? The procedure described above givesthe answer for a <u>single frequency</u>. For a complete answer one has to get the Fourier transform of the shot waveform, carry out the calculation separately for each spectral component, and finally, transform the result back to time domain.

The backward problem has essentially two parts:

(1) how to estimate the parameters of the first layer from the data;

(2) assuming the first layer parameters known, "downward continue" the problem.

The "star product" gives an easy answer to part (2) by noticing that

$$S(0,M) = S(0,1) * S(1,M) \tag{7}$$

or, looking just at S_{21} ,

$$S_{21}(0,M) = M_{21}(1) + M_{22}(1)\, S_{21}(1,M)[I - M_{12}(1)\, S_{21}(1,M)]^{-1} M_{11}(1) \tag{8}$$

where we assume $S_{21}(0,M)$ (the surface data) and $M_{11}(1)$, $M_{12}(1)$, $M_{21}(1)$, $M_{22}(1)$ (the first layer parameters) to be known. Therefore,

$$S_{21}(1,M) = [\, M_{12}(1) + M_{11}(1)[S_{21}(0,M) - M_{21}(1)]^{-1} M_{22}(1)\,]^{-1} \quad (9)$$

Note that this equation gives a different kind of "downward continuation." It gives the result of a new experiment: one in which the first layer has been stripped off, and both the shots and receivers are directly on the second layer. Note again that (9) only gives the answer for a single spectral component.

A similar procedure can be used to upward continue the problem, i.e., adding layers on top, and moving upward both shots and receivers. Such a scheme might be useful, for example, in attempting to convert shallow water data to deep water data, and thus "push down" the multiple reflections.

Reference

1. Redheffer, R., "On the relation of transmission line theory to scattering and transfer", J. Math. Phys., vol. XLI, pp. 1-41, 1962.