L_1 Norm Spectral Estimation

by Luis Canales

Burg (SEP-6) has shown that the average power and the reflection coefficients provide an alternate description of the second order statistics of stationary time series. He also gives a method for the estimation of the sequence of reflection coefficients. The Burg technique consists of finding c_n which minimizes

$$E_2 = \sum_t \{ | e_t^+ + c_n e_t^- |^2 + | e_t^- + c_n e_t^+ |^2 \} \tag{1}$$

where e_t^+ and e_t^- are the error sequences obtained with the forward and backward prediction error filters. An attractive alternative to minimizing (1) is to minimize E_1 defined as follows:

$$E_1 = \sum_t \{ |e_t^+ + c_n e_t^-| + |e_t^- + c_n e_t^+| \} \tag{2}$$

with the idea of obtaining a more robust estimation of the reflection coefficient c_n (Claerbout and Muir, 1973). Note that we are considering only the real case.

Let us for a moment concentrate on minimizing

$$E = \sum_j |y_j - c\, x_j|$$

Claerbout and Muir (1973) have shown that this is equivalent to the weighted median problem and that c has to satisfy

$$\sum_j x_j \operatorname{sign}(y_j - c\, x_j) = 0 \tag{3}$$

We see that each c defines four regions in the x,y plane in such a way that two contribute positive terms and the other two contribute negative terms to the summation (3), as shown in Figure 1.

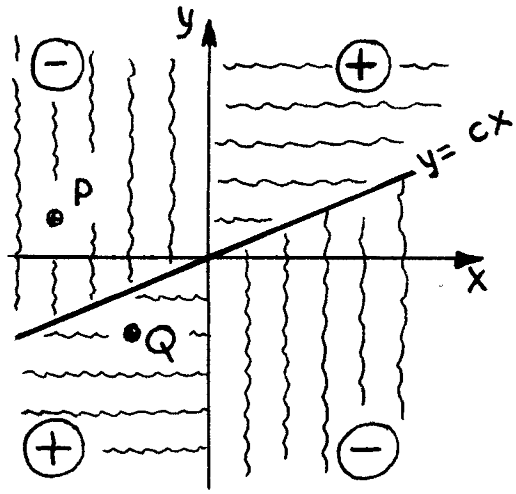

Fig. 1

For instance, point Q has negative error and negative x , therefore, x sign (error) > 0 . On the other hand, point P has negative x and positive error, and thus the contribution is negative.

We see that we can map the left half plane onto the right half plane using the symmetrical projection with respect to the origin. Figure 2 shows the points P and Q projected onto P' and Q' .

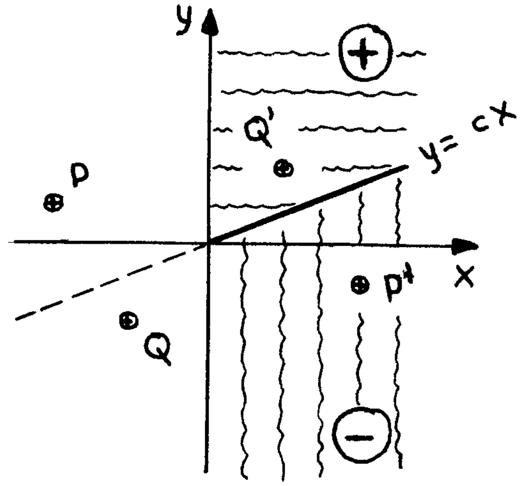

Fig. 2

We see from Figure 2 that points with slope y_i/x_i larger than c are in the positive region and so only the slope and $|x_i|$ are needed to find the terms in the summation (3).

Returning to the original problem, the pairs (x,y) will be of the form (e_t^+, e_t^-) and (e_t^-, e_t^+) so that the line $x=y$ will be an axis of symmetry. In terms of projections onto the right half plane, the +45° and -45° lines are axes of symmetry for points on the first quadrant and on the fourth quadrant respectively, as shown in Figure 3.

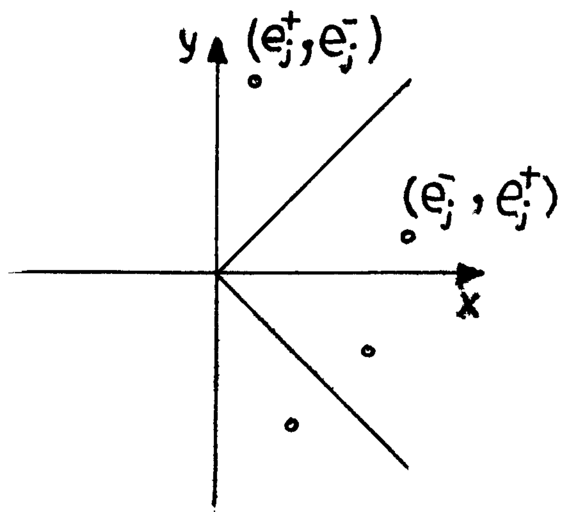

Fig. 3

It is now clear that the solution line should lie between -45° and +45°, because the weights (x's) of points inside that range are larger than the weights of their symmetrical images. Therefore $|c| \leq 1$, as required for the prediction error filter to be minimum phase.

Figure 3 also illustrates another nice property of the L_1 norm in this case. Note that a perfectly correlated set of pairs (x,y) is composed of points lying entirely on the lines at $\pm$ 45°. Points far from those lines will push c towards zero, again due to the larger weights (abscissae) of the points inside $\pm 45°$.

This tendency of pushing the reflection coefficients towards zero means that if a special feature on the time series is unpredictable, the L_1-norm algorithm will not try as hard to predict it as the L_2-norm would. In terms of the spectrum we can say that the L_1-norm will concentrate on the well defined peaks. Those properties of the L_1-norm estimation of the reflection coefficients can be advantageous or disadvantageous according to what we need. Therefore, we have to be careful in the choice of norm.

The fact that the L_1-norm concentrates on the highly predictable part of the time series could be used to advantage for preprocessing before doing Noah deconvolution, because we could get rid of the unpredictable part of the time series.

SUBROUTINE BURGO gives the average power (POWER), the reflection coefficients (C(LA)), the prediction error filter (A(LA)) , the autocorrelation (R(LA)), the prediction variance sequence (V(LA)) and the final prediction error sequence (FPE(LA)) (see Ulrych and Bishop, 1975). If we want to use the Akaike's criterium to stop the recursion, we remove the comment "C" from line 84. X(LX) is the given data and LMAX is the maximum prediction error filter length we are willing to use. If we don't use the Akaikes criterium, then LMAX=LA , otherwise $LA < LMAX$.

SUBROUTINE L1 , is used by BURGO to determine the optimum value of c . It is essentially Hoare's algorithm applied to this particular problem. If LX is larger than 500, then the working arrays SLOPE and IN have to be made larger accordingly.

For completeness we include SUBROUTINE MESPEC, which gives the maximum entropy log-spectrum. It is not the most efficient, but it is very handy for small jobs and can be modified very easily to give the spectrum in a specified frequency band.

Finally, we give an example of a typical calling sequence.

```
REAL  X(50),A(50),R(50),C(20),V(20),FPE(20)
DATA  LX/50/, LMAX/50/
CALL  BURGO(LX,X,LMAX,LA,C,A,R,V,FPE,POWER)
CALL  MESPEC(LX,X,LA,A,R,V(LA))
```

References

Claerbout, J. F., and F. Muir (1973), Robust modeling with erratic data, Geophysics, 38, pp. 826-844.

Ulrych, T. J., and T. N. Bishop (1975), Maximum entropy spectral analysis and autoregressive decomposition, Rev. Geophys. Space Phys., 13, pp. 183-200.

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      SUBROUTINE BURGC(LX,X,LMAX,LA,C,A,R,V,FPE,POWER)
      REAL X(LX),A(LX),R(LX), C(LMAX),V(LMAX),FPE(LMAX)
      C(1)=-1.
      V(1)=1.
      FACTOR = FLOAT(LX-1)/FLOAT(LX+1)
      POWER = X(1)**2
      DO 1 I=2,LX
      A(I-1) = X(I)
      R(I-1) = X(I-1)
    1 POWER = POWER + X(I)**2
      FPE(1) = 1.
      SMALL = 1.
      DO 4 K=2,LMAX
      LIM = LX - K
      CALL LI(LIM+1,A,R,IMIN,CC)
      V(K) = V(K-1) * ( 1. - CC**2 )
      FPE(K) = V(K) * FACTOR * FLOAT(LX + K )/FLOAT(LX-K)
C      IF( FPE(K) .GT. SMALL .OR. C(K) .EQ. 0. ) GO TO 5
      SMALL = FPE(K)
      DO 3 I=1,LIM
      R(I) = R(I) - CC*A(I)
    3 A(I) = A(I+1) - CC * R(I+1)
    4 C(K) = CC
      K = LMAX
    5 LA = K
C
      R(1)=1.
      A(1)=1.
C
      DO 7 JJ=2,LA
      KK=JJ+1
      A(JJ)=0.
      CC=C(JJ)
      RR=V(JJ-1)*CC
      DO 6 I=2,JJ
    6 RR=RR-A(I)*R(KK-I)
      R(JJ)=RR
      JJH=KK/2
      DO 7 I=1,JJH
      HOLD=A(KK-I)
      A(KK-I)=HOLD-CC*A(I)
    7 A(I)=A(I)-CC*HOLD
C
      RETURN
      END
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```
      SUBROUTINE L1 ( N , X , Y, IMIN , T )
      DIMENSION  SLOPE(500), IN(500),X(N),Y(N)
C*******                                  N
C  THIS ROUTINE FINDS  T  SUCH THAT.     SUM ABS(Y(I)+T*X(I)) = MINIMUM
C                                        I=1  + ABS(X(I)+T*Y(I))
C     T = Y(IMIN)/X(IMIN)
C                              IF( N .GT. 500 )  INCREASE THE LENGTH OF
C                                                ARRAYS SLOPE AND IN.
C***********************
C
C                 INITIALIZE CONSTANTS AND ARRAYS
      IS = 0
      SUM = 0.
      KK = N
      DO 1 I=1,N
      IN(I) = I
      HOLD = AMIN1( ABS(X(I)),ABS(Y(I)))
      IF ( HOLD ) 80 , 80 , 90
   80 SLOPE(I) = 0.
      GO TO 1
   90 SLOPE(I) = Y(I) / X(I)
      IF ( ABS( SLOPE(I) ) .GT. 1. ) SLOPE(I) = 1./SLOPE(I)
      SUM = SUM + SIGN( HOLD, SLOPE(I) )
    1 CONTINUE
C              GUESS A CANDIDATE AND  SPLIT NEGATIVE,POSITIVE AND
C                   ZERO CONTRIBUTORS.
    2 IMIN = IN ((KK+1)/2 )
      T = SLOPE (IMIN )
      ZERO = 0.
      DO 7 K=1,KK
      HOLD = AMAX1( ABS( X( IN(K) ) ) , ABS( Y( IN(K) ) ) )
      IF ( SLOPE( IN(K) ) - T ) 4 , 3 , 21
    3 ZERO = ZERO + HOLD
      IN(K) =0
      GO TO 7
    4 IN(K) = - IN(K)
      HOLD = -HOLD
      IF( IS ) 7 , 5 , 6
   21 IF ( IS  ) 6 , 5 , 7
    6 SUM = SUM + HOLD
    5 SUM = SUM + HOLD
    7 CONTINUE
C                      CHECK TO SEE IF WE HAVE THE SOLUTION , OTHERWISE
C                      UPDATE ARRAY IN , WHICH POINTS TO VIABLE CANDIDATES
      SUM = SUM - IS*ZERO
      IF ( ABS( SUM ) .LE. ZERO ) GO TO 10
      IS =  SIGN ( 1. , SUM )
      SUM = SUM - IS*ZERO
      KKK = KK
      KK = 0
      DO 9 K=1 , KKK
      IF ( IN(K) ) 11 , 9 , 12
   11 IF ( IS ) 8 , 9 , 9
   12 IF ( IS ) 9 , 9 , 8
    8 KK = KK + 1
      IN ( KK ) = IABS ( IN(K) )
    9 CONTINUE
      IF ( KK .GE. 1 ) GO TO 2
   10 RETURN
      END
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```
      SUBROUTINE MESPEC(NS,SPEC,M,A,R,V)
      REAL SPEC(NS),A(M),R(M)
C****
C   THIS SUBROUTINE GIVES THE MAXIMUM ENTROPY LOG-SPECTRUM
C***
C   NS=NUMBER OF FREQUENCY POINTS
C   SPEC(NS)= 10*LOG10(SPECTRUM) +CONSTANT   / THE CONSTANT IS
C                        SUCH THAT THE SPECTRUM'S AVERAGE IS =1
C   A(M)= M-LONG PREDICTION ERROR FILTER
C   V= NORMALIZED PREDICTION ERROR VARIANCE (SIGMA**2=V*(AVERAGE-POWER)
C  R(M)=WORK ARRAY IT CONTAINS THE AUTOCORRELATION OF A(M)
C****
      DO 5 I=1,M
      MI=M-I
      RR=0.
      DO 1 J=1,I
    1 RR=RR+A(J)*A(MI+J)
    5 R(MI+1)=RR
C****
C   THE SPECTRUM IS COMPUTED USING THE JENKINS-WATTS ALGORITM
C****
      DBPE=10.*ALOG10(V)
C****
C  NOTE THAT NORMALIZED SPECTRUM = SPECTRUM OF V/A(Z)
C****
      W=3.141593/(NS-1)
      M2=M+2
      DO 3 J=1,NS
      V0=0.
      V1=0.
      C=2.*COS((J-1)*W)
      DO 2 K=2,M
      V2=C*V1-V0+R(M2-K)
      V0=V1
    2 V1=V2
    3 SPEC(J)=DBPE-10.*ALOG10(R(1)+V1*C-V0-V0)
      RETURN
      END
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