Velocity Estimation with Slant Stacks, Part 1

by Jon F. Claerbout

We define a downgoing transformation and its inverse

$$\begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -p & -f \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ z \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} t \\ x \\ z \end{bmatrix} = \begin{bmatrix} 1 & p & f+pb \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} \tag{2}$$

and an upcoming wave transformation and its inverse

$$\begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & -p & f \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ z \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} t \\ x \\ z \end{bmatrix} = \begin{bmatrix} 1 & p & -f-pb \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix} \tag{4}$$

Field data is observed at $z=0$ which means that we can transform field data according to (3) without needing to choose f and b. Only p must be selected. Numerically it is the slant stack stepout. For homogeneous media it may be interpreted as $p = \sin\theta / v$ where v is the yet unknown material velocity and θ is the yet unknown incidence angle. We want to take care not to build into our calculations any assumption of the unknown velocity until the very last step.

Next we will derive an equation for downward continuation following the kind of approach used in "Coupled Slanted Waves, Monochromatic Derivation". The chain rule for partial differentiation gives

$$U_x = U''_{x''} x''_x + U''_{z''} z''_x + U''_{t''} t''_x$$

$$= U''_{x''} - p U''_{t''}$$

$$i k U = (i k'' + i \omega'' p) U'' \quad (5)$$

$$U_z = U''_{z''} z''_z + U''_{x''} x''_z + U''_{t''} t''_z$$

$$= U''_{z''} + b U''_{x''} + f U''_{t''}$$

$$= U''_{z''} + (i k'' b - i \omega'' f) U'' \quad (6)$$

$$U_t = U''_{t''} = - i \omega'' U'' \quad (7)$$

Locally the cosine of the angle ϕ of wave propagation direction from the normal is

$$\cos\phi = (1 - \frac{k^2 v^2}{\omega^2})^{1/2}$$

$$= [1 - \frac{(k'' + \omega'' p)^2 v^2}{\omega''^2}]^{1/2}$$

$$= [1 - p^2 v^2 - 2 p v (\frac{k'' v}{\omega''}) - (\frac{k'' v}{\omega''})^2]^{1/2}$$

$$= (1 - p^2 v^2)^{1/2} [1 - \frac{2 pv}{1-p^2 v^2} (\frac{k'' v}{\omega''}) - \frac{1}{1-p^2 v^2} (\frac{k'' v}{\omega''})^2]^{1/2} \quad (8)$$

To order ε^2 one has $(1 + a\varepsilon + b\varepsilon^2)^{1/2} = 1 + (a/2)\varepsilon + (b/2 - a^2/8)\varepsilon^2$.
So to second order in $k''v/\omega''$ equation (8) becomes

$$\cos\phi = (1 - p^2v^2)^{1/2}\left[1 - \frac{pv}{1-p^2v^2}\left(\frac{k''v}{\omega''}\right) - \frac{1}{2(1-p^2v^2)^2}\left(\frac{k''v}{\omega''}\right)^2\right] \quad (9)$$

Now we write the local wave equation

$$U_z = -\frac{i\omega}{v}\cos\phi \; U \quad (10)$$

Substituting (6), (7) and (9) into (10) we get

$$U''_{z''} + (i k'' b - i \omega'' f) U'' = \quad (11)$$

$$= -\frac{i\omega''}{v}\left[(1-p^2v^2)^{1/2} - \frac{pv}{(1-p^2v^2)^{1/2}}\frac{k''v}{\omega''} - \frac{1}{2(1-p^2v^2)^{3/2}}\left(\frac{k''v}{\omega''}\right)^2\right] U''$$

If we assert that the velocity is homogeneous, even though we do not know its value, then we can pick b and f to simplify (11) a great deal. Taking

$$f = \frac{(1 - p^2v^2)^{1/2}}{v} \quad (12)$$

$$b = \frac{pv}{(1 - p^2v^2)^{1/2}} \quad (13)$$

we find that (11) reduces to

$$U''_{z''} = -\frac{1}{2(1-p^2v^2)^{3/2}}\frac{(ik'')^2}{-i\omega''}U''$$

or

$$U''_{z''t''} = -\frac{v}{2(1-p^2v^2)^{3/2}}U''_{x''x''} \quad (14)$$

Now we expect to downward continue upcoming waves with (14) to various depths and compare to the downgoing waves. But (14) seems to imply that we need the yet unknown velocity v. Actually the situation is not as bad as it seems. The zero order effect of velocity is the shifting terms in the transformation. The role of velocity in (14) is to do diffraction or shifts which are second order in dip $k'v/\omega'$. In practice one may say that a rough guess at the velocity will do. A more exacting approach with little extra computational effort is to note that a change of velocity in (14) is identical to a rescaling of the z'' axis. Thus, once having done the work to project U'' down through the earth for one velocity, a mere rescaling of the z'' axis gives other velocities.

Now that we have the upcoming wave U'' down at any depth for any velocity the job is to fit it to a downgoing wave. The downgoing wave can be calculated with varying degrees of accuracy. The simplest, which we will consider first, is to assume the medium has a constant but unknown velocity. Let $S(t)$ denote some shot waveform. An expression for a downgoing plane wave carrying this waveform is

$$D(t,x,z) = S\left(t - z\frac{\cos\theta}{v} - x\frac{\sin\theta}{v}\right) \qquad (15)$$

The downgoing wave which corresponds to our slant stacked seismic data is that for which $\sin\theta / v = p$.

$$D(t,x,z) = S\left(t - z(1-p^2v^2)^{1/2}/v - xp\right) . \qquad (16)$$

Now we choose any one of three coordinate systems in which to fit D to U. Let us use the frame in which U is calculated. From (4) and (16) we have

$$D(t,x,z) = D''(t'',x'',z'') =$$

$$= S[\, t'' + p\,x'' - (f+pb)\,z'' - z''\,\frac{(1-p^2v^2)^{1/2}}{v} - (x''-bz'')\,P\,]$$

$$= S[\, t'' - z''\,(\, f + \frac{(1-p^2v^2)^{1/2}}{v}\,)\,]\ ;$$

using (12),

$$D''(t'',x'',z'') = S\,(\, t'' - \frac{2(1-p^2v^2)^{1/2}}{v}\, z''\,) \qquad (17)$$

Ordinarily we would hope to have a shot waveform which approximates a delta function so that the downgoing wave would be a pulse when the argument of S vanishes, that is, when

$$t'' = 2\,(\,1 - p^2\,v^2\,)^{1/2}\ z''/v \qquad (18a)$$

which in terms of cartesian variables from (3) is when

$$t - p\,x + f\,z = 2\,(1 - p^2\,v^2\,)^{1/2}\ z/v \qquad (18b)$$

Neglect for the moment angular dependence of reflection coefficients and angular dependence of source and geophone antenna radiation patterns. Let t_d denote the arrival time of the downgoing wave. Then the reflectivity model should be

$$c\,(x,z) = U\,(\,t_d,\ x,\ z\,) \qquad (19a)$$

If we restrict t'' to be related to z'' by (18a) then the reflectivity could also be written as

$$c(x,z) = U''(t'', x'', z'') \tag{19b}$$

We have seen that U'' is slowly variable with z'' so that z'' in (19b) can be replaced by an approximation, namely (18a) with a preliminary rough velocity estimate $\bar{v}$. This achieves the practical result of approximating U by a function $\hat{U}$ of two variables only, namely

$$\begin{aligned} c(x,z) &= U''[t'', x'', .5\,\bar{v}\,t''(1-p^2\bar{v}^2)^{-1/2}] \\ &= \hat{U}''(t'', x'') \end{aligned} \tag{20}$$

Using $pv = \sin\theta$, (18a), (13), and (3) we express (20) in Cartesian variables

$$c(x,z) = \hat{U}''\left(\frac{2z\cos\theta}{v}, x + z\tan\theta\right) \tag{21}$$

To determine the velocity function v we can use various fitting techniques, all based on the idea that $\hat{U}''$ can be computed for various values of p . The correct velocity v is determined at any x and z such that $c(x,z)$ comes closest to being independent of p . One technique vaguely similar to current industry practice would be to scan v looking for a maximum of the sum

$$\max_{v} \left| \sum_{p} \hat{U}\left(\frac{2z\cos\theta}{v}, x + z\tan\theta\right) \right| \tag{22}$$

Obviously summation gates can also be used on x and z as for example

$$\max_{v} \sum_{x,z} \left(\sum_{p} \hat{U} \right)^2 \tag{23}$$

Presumably the velocity v determined in this way is in some way related to the familiar rms velocity. A more detailed analysis will be required to express this idea more precisely. We can allow for angle dependent antenna effects by a phaseless or a minimum phase spectral balancing. If the spectral balancing is time variable then we are also compensating for angle dependent reflection coefficients.

Now let us go on to study lateral velocity variations. Then we will need also to have an equation for the downgoing wave. It will be derived from an equation like (10), namely

$$D_z = \frac{i\omega}{v} \cos\phi \quad D \tag{24}$$

The derivation is like that following (10) except that b and f have the opposite sign because we use (1) and (2) rather than (3) and (4). The lowest approximation is to neglect dip dependent terms, that is, terms containing $k''v / \omega''$. Thus a calculation like (11) leads to

$$D'_{z'} = [\frac{i\omega''(1 - p^2v^2)^{1/2}}{v} - i\,\omega''\, f \,] D'$$

$$= i\,\omega'' [\frac{(1 - p^2v^2)^{1/2}}{v} - \frac{(1 - p^2\bar{v}^2)^{1/2}}{\bar{v}}] D'$$

If $v \simeq \bar{v}$ then

$$D'_{z'} = - D'_{t'} [(v-\bar{v}) \left| \frac{\partial}{\partial v} \frac{(1 - p^2v^2)^{1/2}}{v} \right|_{\bar{v}}]$$

$$= - D'_{t'} [\frac{v-\bar{v}}{\bar{v}^2} \frac{1}{(1 - p^2\bar{v}^2)^{1/2}}]$$

Write this as

$$D'_{z'} = -D'_{t'} \quad S(x',z')$$

where S is some slowness function. The solution to this will merely be some time shift which is a line integral of the slowness function from $z'=0$ to any z' along a path of constant x'. In (x,z) space this is the path of the downgoing wave. Thus, the inclusion of this type of term will relate line integrals of slowness along the "frame path" to observed time shifts. From these integrals one must obtain "interval velocities" keeping in mind the lateral separation of the up and downgoing paths.