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Coupled Slanted Waves, Monochromatic Derivation

by Jon F. Claerbout

This is another derivation of the equations coupling up and downgoing waves when both move with a substantial horizontal component. We begin with Estevez' upcoming wave transformation, SEP-5-p. 21.

$$x' = x + z \tan\theta \tag{1a}$$

$$z' = z \tag{1b}$$

$$t' = t + \frac{z}{v}\cos\theta - \frac{x}{v}\sin\theta \tag{1c}$$

and the statement that

$$U(x, z, t) = U'(x', z', t') \tag{2}$$

and the chain rule for derivatives

$$U_x = U'_{x'}\, x'_x + U'_{z'}\, z'_x + U'_{t'}\, t'_x = U'_{x'} - U'_{t'} \sin\theta / v \tag{3a}$$

$$U_z = U'_{x'}\, x'_z + U'_{z'}\, z'_z + U'_{t'}\, t'_z = U'_{x'} \tan\theta + U'_{z'} + U'_{t'} \cos\theta / v \tag{3b}$$

$$U_t = U'_{x'}\, x'_t + U'_{z'}\, z'_t + U'_{t'}\, t'_t = U'_{t'} \tag{3c}$$

Fourier transformation of x' and t' will give monochromatic solutions like

$$\exp(-i\omega' t' + i k' x') \tag{4}$$

From (3) fourier transformation gives

$$i k U = (i k' + i \omega' \sin\theta / v)\, U' \tag{5a}$$

$$U_z = U'_{z'} + (i k' \tan\theta - i \omega' \cos\theta / v)\, U' \tag{5b}$$

$$-i\omega U = -i\omega' U' \tag{5c}$$

The cosine of the angle ϕ of wave propagation direction from the normal is

$$\cos\phi = \left(1 - \frac{k^2 v^2}{\omega^2}\right)^{1/2}$$

$$= \left[1 - \left(\frac{k'v + \omega'\sin\theta}{\omega'}\right)^2\right]^{1/2}$$

$$= \left[1 - \sin^2\theta - 2\frac{k'v}{\omega'}\sin\theta - \left(\frac{k'v}{\omega'}\right)^2\right]^{1/2}$$

$$= \cos\theta\left[1 - 2\frac{k'v}{\omega'}\frac{\sin\theta}{\cos^2\theta} - \left(\frac{k'v}{\omega'\cos\theta}\right)^2\right]^{1/2} \qquad (6)$$

To order ε^2 one has $(1 + a\varepsilon + b\varepsilon^2)^{1/2} = 1 + (a/2)\varepsilon + (b/2 - a^2/8)\varepsilon^2$, so to second order in $k'v/\omega'$ equation (6) becomes

$$\cos\phi = \cos\theta - \frac{k'v}{\omega'}\tan\theta - \left(\frac{k'v}{\omega}\right)^2\frac{1}{2\cos^3\theta} \qquad (7)$$

and the inverse of $\cos\phi$ to first order is

$$1/\cos\phi = \frac{1}{\cos\theta}\left(1 + \frac{k'v}{\omega'}\frac{\sin\theta}{\cos^2\theta}\right) \qquad (8)$$

The basic monochromatic coupling equation for stratified media (from my book, McGraw Hill, 1976) is

$$U_z = -\frac{i\omega}{v}\cos\phi\ U - \frac{1}{2}\frac{I_z}{I}D \qquad (9)$$

where the impedance I is given by

$$I = \rho v/\cos\phi \qquad (10)$$

Inserting (5b), (5c), and (7) into (9) we get

$$U'_{z'} + (i\,k'\tan\theta - i\omega'\cos\theta/v)U' = -\frac{i\omega'}{v}\left(\cos\theta - \frac{k'v}{\omega'}\tan\theta - \left(\frac{k'v}{\omega'}\right)^2\frac{1}{2\cos^3\theta}\right)U' - \frac{1}{2}\frac{I_z}{I}D$$

$$U'_{z'} = -\frac{1}{2\cos^3\theta}\frac{(ik')^2}{(-i\omega')}U' - \frac{1}{2}\frac{I_z}{I}D \qquad (11)$$

Differentiating (10) for insertion into (11) and using (8) we get to first order in $k'v/\omega'$

$$I_z / I = \frac{(\rho v)_z / \cos\phi + \rho v \frac{k'}{\omega'} \sin\theta \, v_z / \cos^3\theta}{\rho v / \cos\phi}$$

$$= \frac{(\rho v)_z}{\rho v} + \frac{k'}{\omega'} \frac{\sin\theta}{\cos^2\theta} v_z \qquad (12)$$

So (11) with (12) inverse transformed gives the final result

$$U'_{z'} = -\frac{v}{2\cos^3\theta} U'^{t'}_{x'x'} - \frac{1}{2} \frac{(\rho v)_z}{\rho v} D - \frac{1}{2} v_z \frac{\sin\theta}{\cos^2\theta} D^{t'}_{x'} \qquad (13)$$

A curious result of this derivation is that the ∂_{xz} term of Estevez' derivation does not appear. This implies that it is of the same order as the Fresnel approximation. The comparatively unfamiliar term $v_z \frac{\sin\theta}{\cos^2\theta} D^{t'}_{x'}$ provides the angular dependence of reflection coefficient magnitude. At this point in time we can only wonder whether the new term will be significant in dealing with field data.