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Slanted Multiple Reflection Calculation

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To the accuracy of the Noah approximation, Figure 1 shows the upcoming wave u_3 which arrives at the surface at time $t=3$ as a function of earlier waves downgoing from the surface, namely d_0 , d_1 , and d_2 .

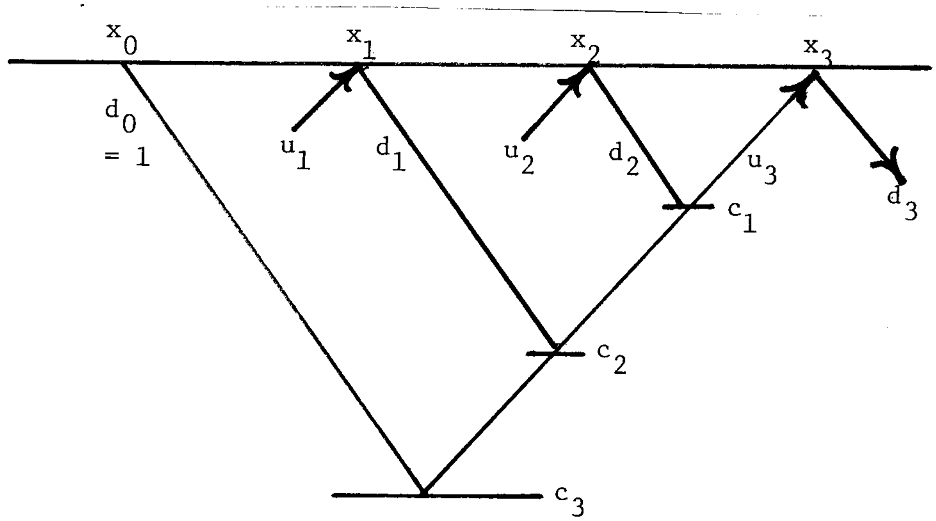


Fig. 1. Noah wave geometry.

We have

$$u_3 = c_3 d_0 + c_2 d_1 + c_1 d_2 \tag{1}$$

Using the initial condition $d_0 = 1$ and the surface reflection condition $d_t = - u_t$ we get

$$u_3 = c_3 - c_2 u_1 - c_1 u_2 \tag{2}$$

In order to work in a slant frame we merely attach an x" functional dependence to each of the waves and reflectors. By comparing Fig. 1

to equation (3) you will note that this is done in the skewed, natural frame for upcoming waves.

$$u_3(x'') = c_3(x'') - c_2(x'')u_1(x''-2) - c_1(x'')\, u_2(x''-1) \qquad (3)$$

The subscript on c is a time-depth subscript, say t'' or z'' and the subscript on u is slant upcoming wave time t''. Thus, in general

$$u_{t''}(x'') = c_{t''}(x'') - \sum_{z''=1}^{t''-1} c_{z''}(x'')\, u_{t''-z''}\,(x''-z'') \qquad (4)$$

Note that (4) can be used recursively to find u from c for increasing x'' and increasing t''. To find c from u the same equation may be used. A program and some examples are in Figures 2, 3 and 4.

The inner loop in the computer program is

```
U(IX,IT)  =  U(IX,IT) - C(IX,IZ) * U(IX-IZ+1,IT-IZ+1)          (5)
```

Recalling that the right hand U really is the negative of the downgoing wave at the surface we have

```
U(IX,IT) = U(IX,IT) + C(IX,IZ) * D(IX-IZ+1,IT-IZ+1)             (6)
```

The iteration (6) may be regarded as the extrapolation of a difference equation for the upcoming wave. It starts at zero at great depth and as we extrapolate upward we accumulate the source terms $c(z)\, D(z)$ all the way up to the surface, say

$$U\,(z - \Delta z) = U(z) + \Delta z\; c(z)\, D(z)$$

or in partial differential form

$$\frac{\partial U''}{\partial z''} = - c\, D \qquad (7)$$

```
$WATFIV
C          TEST SLANT NOAH WITHOUT DIFFRACTIONS
           DIMENSION U(52,10),C(52,10),D(52,10)
           NX=52
           NT=10
           DO 10 IX=1,NX
           DO 10 IT=1,NT
    10     C(IX,IT)=0.
           DO 20 IX=1,20
    20     C(IX,3)=.8
           DO 25 IX=40,NX
    25     C(IX,3)=.8
           DO 30 IX=1,NX
    30     C(IX,6)=.1
           DO 40 IX=1,NX
           D(IX,1)=1.
           DO 40 IT=2,NT
    40     D(IX,IT)=0.
           CALL NOAHS1(-1.,NX,NT,C,U)
           CALL OUT(NX,NT,U)
           CALL NOAHS1(1.,NX,NT,U,C)
           CALL OUT(NX,NT,C)
           STOP
           END


           SUBROUTINE NOAHS1(SIGN,NX,NT,C,U)
           DIMENSION U(NX,NT),C(NX,NT)
           DO 10 IT=1,NT
           DO 10 IX=1,NX
    10     U(IX,IT)=C(IX,IT)
           DO 20 IT=3,NT
           NZ=IT-1
           DO 20 IZ=2,NZ
           DO 20 IX=IT,NX
    20     U(IX,IT)=U(IX,IT)+C(IX,IZ)*U(IX-IZ+1,IT-IZ+1)*SIGN
           RETURN
           END


           SUBROUTINE OUT(NX,NT,C)
           DIMENSION  C(NX,NT),LINE(100)
           PRINT 10
    10     FORMAT('1 NEXT SECTION')
           DO 30 IX=1,NX
           DO 20 IT=1,NT
    20     LINE(IT)=100.5*C(IX,IT)
    30     PRINT 40,IX,(LINE(IT),IT=1,NT)
    40     FORMAT(20X,12I6)
           RETURN
           END
```

Fig. 2. Slanted Noah Multiples Program.

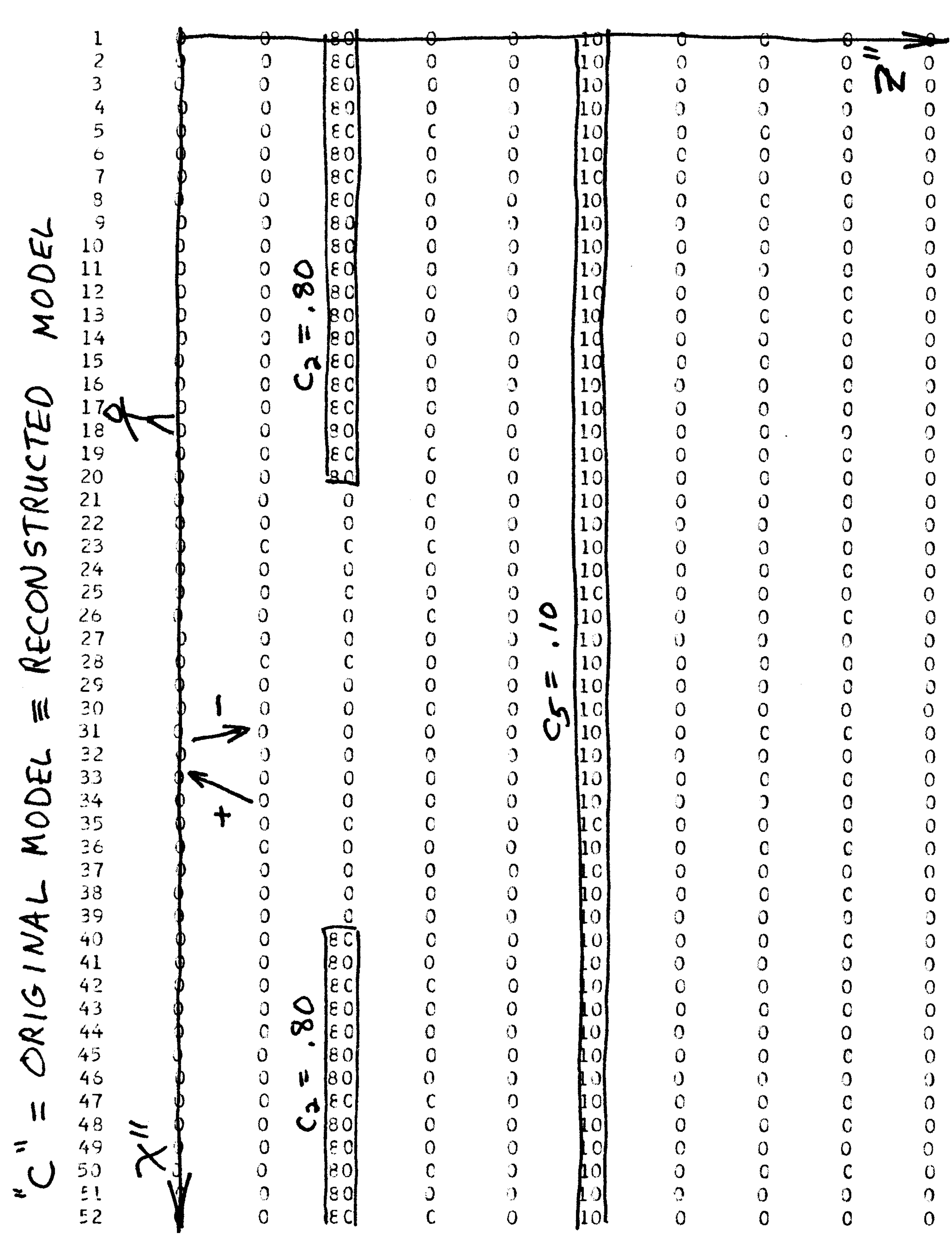


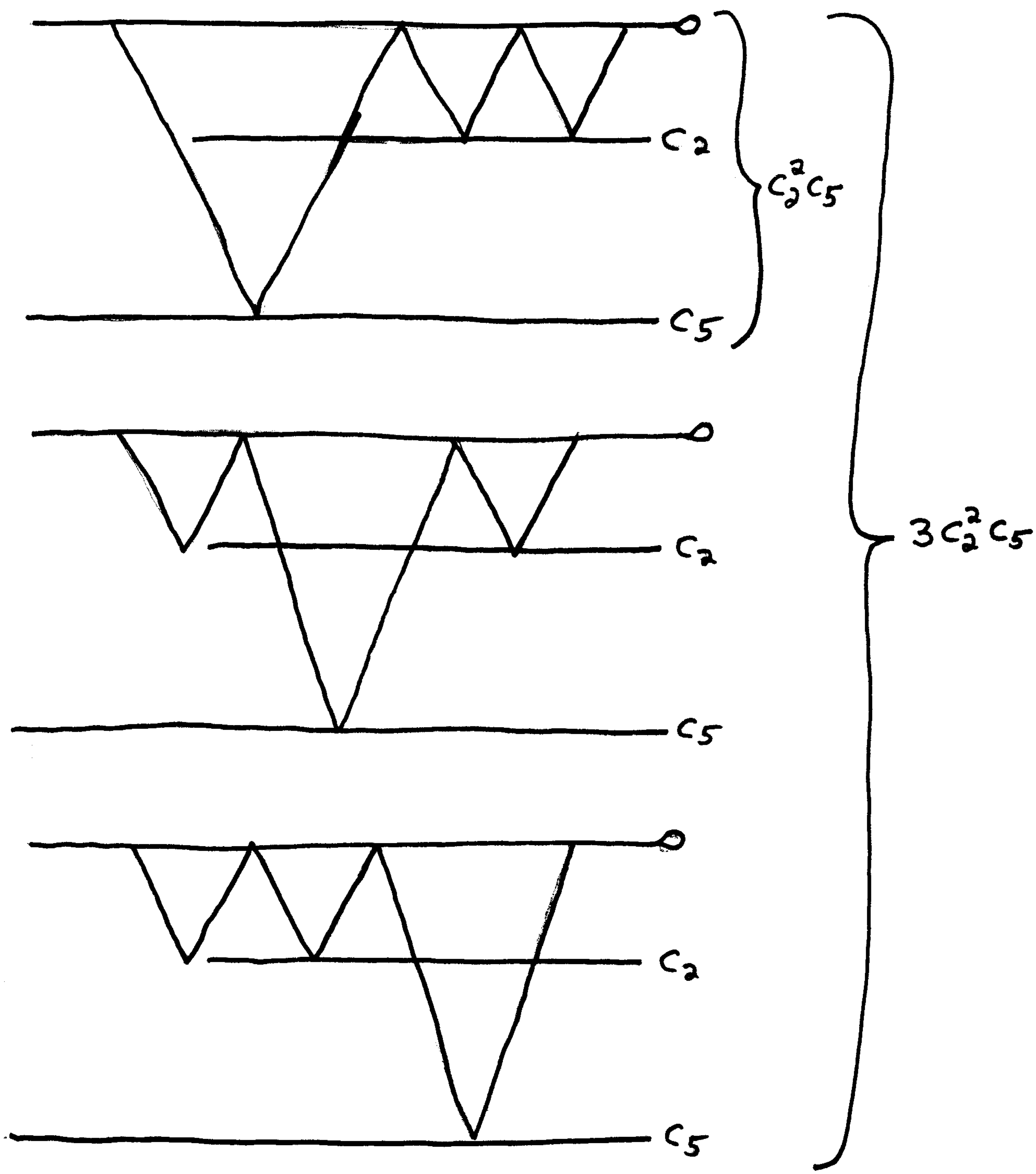
Fig. 3. Earth Model Test Case.

"U" = UPCOMING WAVES times 100

x"										t"
1	0	0	80	0	0	10	0	0	0	0
2	0	0	80	0	0	10	0	0	0	0
3	0	0	80	0	0	10	0	0	0	0
4	0	0	80	0	0	10	0	0	0	0
5	0	0	80	0	-64	10	0	0	0	0
6	0	0	80	0	-64	10	0	0	0	0
7	0	0	80	0	-64	10	51	0	0	0
8	0	0	80	0	-64	10	51	-16	0	0
9	0	0	80	0	-64	10	51	-16	-41	0
10	0	0	80	0	-64	10	51	-16	-41	19
11	0	0	80	0	-64	10	51	-16	-41	19
12	0	0	80	0	-64	10	51	-16	-41	19
13	0	0	80	0	-64	10	51	-16	-41	19
14	0	0	80	0	-64	10	51	-16	-41	19
15	0	0	80	0	-64	10	51	-16	-41	19
16	0	0	80	0	-64	10	51	-16	-41	19
17	0	0	80	0	-64	10	51	-16	-41	19
18	0	0	80	0	-64	10	51	-16	-41	19
19	0	0	80	0	-64	10	51	-16	-41	19
20	0	0	80	0	-64	10	51	-16	-41	19
21	0	0	0	0	0	10	0	-8	0	6
22	0	0	0	0	0	10	0	-8	0	6
23	0	0	0	0	0	10	0	-8	0	6
24	0	0	0	0	0	10	0	-8	0	6
25	0	0	0	0	0	10	0	-8	0	6
26	0	0	0	0	0	10	0	0	0	0
27	0	0	0	0	0	10	0	0	0	0
28	0	0	0	0	0	10	0	0	0	0
29	0	0	0	0	0	10	0	0	0	0
30	0	0	0	0	0	10	0	0	0	0
31	0	0	0	0	0	10	0	0	0	0
32	0	0	0	0	0	10	0	0	0	0
33	0	0	0	0	0	10	0	0	0	0
34	0	0	0	0	0	10	0	0	0	0
35	0	0	0	0	0	10	0	0	0	0
36	0	0	0	0	0	10	0	0	0	0
37	0	0	0	0	0	10	0	0	0	0
38	0	0	0	0	0	10	0	0	0	0
39	0	0	0	0	0	10	0	0	0	0
40	0	0	80	0	0	10	0	-8	0	0
41	0	0	80	0	0	10	0	-8	0	0
42	0	0	80	0	-64	10	0	-8	0	6
43	0	0	80	0	-64	10	0	-8	0	6
44	0	0	80	0	-64	10	51	-8	0	6
45	0	0	80	0	-64	10	51	-16	0	6
46	0	0	80	0	-64	10	51	-16	-41	6
47	0	0	80	0	-64	10	51	-16	-41	19
48	0	0	80	0	-64	10	51	-16	-41	19
49	0	0	80	0	-64	10	51	-16	-41	19
50	0	0	80	0	-64	10	51	-16	-41	19
51	0	0	80	0	-64	10	51	-16	-41	19
52	0	0	80	0	-64	10	51	-16	-41	19

start up transient; C_2; $-C_2^3$; C_2^3; $-2C_2C_5$; $-C_2^4$; $3C_2^2C_5$; $-C_2C_5$; $C_2^2C_5$; C_5; see next figure

Fig. 4. Upcoming Waves in Test Case.



see previous figure

Fig. 5. The cutoff of two peglegs at a terminating reflector.

A reason to use partial derivative notation rather than ordinary differentiation is that the equation is integrated along constant x'' , not constant x . There is of course a companion equation which expresses our concept that the downgoing wave does not change on its trip from the free surface to the reflectors. It is

$$\frac{\partial}{\partial z'} D' = 0 \tag{8}$$

Now let us go from the one dimensional problem which we have been doing to the two dimensional problem. Obviously (7) and (8) must be replaced by the final results of "Coupled Slanted Beams, Equations for Multiple Program", which from page 35 are

$$\partial_{z'} D'(x',z',t') = \frac{v}{2\cos^3\theta} \partial^{t'}_{x'x'} D(x',z',t') \tag{9}$$

$$\partial_{z''} U''(x'',z'',t'') = \frac{-v}{2\cos^3\theta} \partial^{t''}_{x''x''} U''(x'',z'',t'') - c''(x'',z'')\, D'\left(x''-2z''\tan\theta, z'', t''-\frac{2z''}{v}\cos\theta\right) \tag{10}$$

Equation (9) obviously generalizes (8) and (10) obviously generalizes (7). Note by comparison of (10) to (6) that the one dimensional calculation which we did implies that

$$\Delta t = \frac{2}{v}\cos\theta \quad \Delta z \tag{11a}$$

$$\Delta x = 2\tan\theta \quad \Delta z \tag{11b}$$

Naturally program modifications will be needed to accommodate most field data.

Let us now consider where the diffraction and migration terms ∂^t_{xx} need to be introduced into the one dimensional program. As a first step we rewrite the computer program of Figure 2 in the form of the program in Figure 6. First of all, ignore the indented cards which do the diffractions. The rest of the program will produce exactly the same result as Figure 4. The new program has separate grids for U and D. The initialization and surface boundary conditions appear more explicitly. The initial downgoing wave may now be an arbitrary waveform, not just an impulse. Also, the integration of $c\,D$ is now done from the reflectors up toward the surface (rather than the other way around). Figure 7 shows the results of using the indented cards which calculated diffraction terms. Naturally, it is difficult if not impossible to determine whether the result is plausible without a full blown 8 pt/wavelength test.

The objective of this presentation is to make a clear, uncluttered exposition of a complicated calculation. The explicit method was selected for diffraction. Of the two forms $U_{zt} = -U_{xx} - c\,D_t$ and $U_z = -U^t_{xx} - c\,D$, the second, which we have called the integrated form, was selected. The integrated form is nice in this problem because in the other form the coupling $c\,D_t$ straddles two time points thereby confusing matters.

Since there is no earlier SEP documentation of an explicit, integrated form, the program derivation is now presented. Define

$$4\,a = \frac{v}{2}\,\frac{\Delta z\;\Delta t}{\Delta x^2} \tag{12}$$

The differential equation

$$P_{zt} = \frac{v}{2}\,P_{xx} \tag{13}$$

```
C          TEST SLANT NOAH IN TWO DIMENSIONS
           DIMENSION U(52,10),C(52,10),D(52,10)
           NX=52
           NT=10
           DO 10 IX=1,NX
           DO 10 IT=1,NT
   10      C(IX,IT)=0.
           DO 20 IX=1,20
   20      C(IX,3)=.8
           DO 25 IX=40,NX
   25      C(IX,3)=.8
           DO 30 IX=1,NX
   30      C(IX,6)=.1
           DO 40 IX=1,NX
           D(IX,1)=1.
           DO 40 IT=2,NT
   40      D(IX,IT)=0.
           CALL NOAHS2(NX,NT,C,D,U)
           CALL OUT(NX,NT,U)
           STOP
           END
```

```
           SUBROUTINE NOAHS2(NX,NT,C,D,U)
           DIMENSION U(NX,NT),C(NX,NT),D(NX,NT)
               DIMENSION DL(52,10),A(52),AP(52),SU(52,10),SD(52,10)
           DO 10 IX=1,NX
   10      U(IX,1)=0.
           DO 50 IT=2,NT
           DO 20 IX=1,NX
               A(IX)=0.
               SU(IX,IT)=0.
               SD(IX,IT)=0.
               DL(IX,1)=D(IX,1)
               DL(IX,IT)=D(IX,IT)
   20      U(IX,IT)=0.
           DO 40 KT=2,IT
           IZ=IT-KT+2
               CALL MOVE(NX,DL(1,KT),AP)
               CALL MOVE(NX,D(1,KT),DL(1,KT))
               CALL DIFF(NX,SD(1,KT),A,D(1,KT),DL(1,KT-1),D(1,KT))
               CALL MOVE(NX,AP,A)
           DO 30 IX=IZ,NX
   30      U(IX,IT)=U(IX,IT)+C(IX,IZ)*D(IX-IZ+1,IT-IZ+1)
               CALL MOVE(NX,U(1,IT-1),AP)
               CALL DIFF(NX,SU(1,KT),A,U(1,IT),U(1,IT-1),U(1,IT))
               CALL MOVE(NX,AP,A)
   40      CONTINUE
           DO 50 IX=1,NX
   50      D(IX,IT)=D(IX,IT)-U(IX,IT)
           RETURN
           END
```

(first half of Fig. 6)

```
      SUBROUTINE DIFF(NX,SUM,Q1,Q2,P1,P2)
      DIMENSION SUM(NX),Q1(NX),Q2(NX),P1(NX),P2(NX),T(52),S(52)
      DO 10 IX=1,NX
      SUM(IX)=SUM(IX)+Q2(IX)+Q1(IX)+2.*P1(IX)
   10 S(IX)=SUM(IX)+Q2(IX)
      NX1=NX-1
      DO 20 IX=2,NX1
   20 T(IX)=S(IX-1)-2.*S(IX)+S(IX+1)
      DO 30 IX=2,NX1
   30 P2(IX)=Q2(IX)+.125*T(IX)
      P2(1)=P2(2)
      P2(NX)=P2(NX1)
      RETURN
      END
```

```
      SUBROUTINE MOVE(NX,A,B)
      DIMENSION A(NX),B(NX)
      DO 10 IX=1,NX
   10 B(IX)=A(IX)
      RETURN
      END
```

Fig. 6. Program for synthesis of slant diffracted multiples. When the indented cards are removed the output is the 1.5 dimensional model (interlocking one-dimensional models) of Figure 4. The output of this program, the two-dimensional diffracted multiples is in Figure 7.

TION

1	0	0	30	20	-14	-17	6	16	-1	-15
2	0	0	30	20	-14	-17	6	16	-1	-15
3	0	0	55	-2	-16	-20	26	-1	8	-33
4	0	0	75	-11	-30	-13	32	1	-7	-19
5	0	0	80	-5	-47	2	19	21	-38	16
6	0	0	80	0	-56	14	12	21	-40	19
7	0	0	80	0	-61	19	23	-3	-15	-11
8	0	0	80	0	-63	16	40	-24	-2	-22
9	0	0	80	0	-64	12	50	-28	-10	-5
10	0	0	80	0	-64	10	52	-24	-24	16
11	0	0	80	0	-64	10	52	-20	-35	30
12	0	0	80	0	-64	10	51	-17	-40	32
13	0	0	80	0	-64	9	51	-15	-41	27
14	0	0	80	0	-64	9	51	-14	-41	20
15	0	0	80	0	-64	8	52	-13	-43	15
16	0	0	80	0	-65	8	54	-13	-47	16
17	0	0	80	0	-68	11	58	-17	-51	25
18	0	0	80	-5	-69	19	57	-27	-47	35
19	0	0	75	-12	-59	25	49	-35	-32	34
20	0	0	55	-7	-41	20	33	-27	-18	22
21	0	0	25	7	-21	7	17	-9	-12	8
22	0	0	5	12	-4	0	5	0	-5	0
23	0	0	0	5	4	2	-1	0	1	-3
24	0	0	0	0	4	9	-3	-2	3	-1
25	0	0	0	0	1	11	-2	-4	2	1
26	0	0	0	0	0	11	0	-4	0	3
27	0	0	0	0	0	10	0	-3	-1	4
28	0	0	0	0	0	10	0	-1	-2	3
29	0	0	0	0	0	10	0	0	-2	1
30	0	0	0	0	0	10	0	0	-1	0
31	0	0	0	0	0	10	0	0	-1	-1
32	0	0	0	0	0	10	0	0	0	-1
33	0	0	0	0	0	10	0	0	0	0
34	0	0	0	0	0	11	0	0	0	0
35	0	0	0	0	1	12	0	0	0	0
36	0	0	0	0	4	10	-1	0	0	0
37	0	0	0	5	4	8	-2	0	0	0
38	0	0	5	12	0	6	-4	3	-2	0
39	0	0	25	7	-5	1	2	0	1	-6
40	0	0	55	-7	-9	-5	15	-9	9	-18
41	0	0	75	-12	-24	1	14	-4	-3	-4
42	0	0	80	-5	-44	17	5	10	-26	22
43	0	0	80	0	-56	24	9	5	-25	17
44	0	0	80	0	-61	23	26	-15	-4	-10
45	0	0	80	0	-63	17	43	-31	2	-16
46	0	0	80	0	-64	13	51	-32	-10	5
47	0	0	80	0	-64	10	53	-25	-27	27
48	0	0	80	0	-64	10	52	-20	-38	36
49	0	0	80	0	-64	10	52	-17	-43	34
50	0	0	80	0	-64	10	51	-16	-43	28
51	0	0	80	0	-64	10	51	-15	-43	22
52	0	0	80	0	-64	10	51	-15	-43	22

Fig. 7. Upcoming waves of Figure 4 with diffractions.

is then given for $b = 1/12$ to 4^{th} order accuracy in x by the difference equation

$$\delta_t \, \delta_z \, P = 4a \frac{\delta_{xx}}{1 + b\,\delta_{xx}} P \tag{14}$$

We have the bilinear transform

$$\delta_t = \frac{2(1-Z)}{(1+Z)} = 2s \tag{15}$$

which reduces (14) to

$$(1 + b\,\delta_{xx})\,\delta_z P = \frac{2a}{s}\,\delta_{xx} P \tag{16}$$

Writing the z-difference as a superscript we have

$$(1 + b\,\delta_{xx})(P^n - P^{n-1}) = \frac{a}{s}\,\delta_{xx}(P^n + P^{n-1}) \tag{17}$$

Bring P^n to the left

$$[\,1 + \delta_{xx}(b - \frac{a}{s})\,]\,P^n = P^{n-1} + \delta_{xx}(b + \frac{a}{s})\,P^{n-1} \tag{18}$$

Note that $(1/s - 1) = 2Z\,/\,(1-Z)$ and add the following identity to (18)

$$\delta_{xx}\left(\frac{a}{s} - a\right) P^n = \delta_{xx}\, a \frac{2Z}{1-Z} P^n \tag{19}$$

getting

$$[\,1 + \delta_{xx}(b-a)\,]\,P^n = P^{n-1} + \delta_{xx}\left[\,b\,P^{n-1} + \frac{1+Z}{1-Z}\,a\,P^{n-1} + a\frac{2Z}{1-Z}P^n\,\right] \tag{20}$$

Now we choose Δz such that $a=b$ or we drop 4^{th} order accuracy (see SEP-2 p. 109) and choose b equal a , even though a may not be $1/12$, subject to $b=a < 1/4$ for stability. Then

$$P^n = P^{n-1} + a\,\delta_{xx} \{ P^{n-1} + \frac{1}{1-Z} [(1+Z)\, P^{n-1} + 2Z\, P^n] \} \qquad (21)$$

Now considering P^n and P^{n-1} to be Z-transform polynomials as $p_0 + p_1 Z + p_2 Z^2 + \ldots$ and identifying the coefficient of Z^t in (21) we get the time domain equation which was the basis for subroutine DIFF in Figure 6.

$$p_t^n = p_t^{n-1} + a\,\delta_{xx} [p_t^{n-1} + \sum_{k=-\infty}^{t} (p_k^{n-1} + p_{k-1}^{n-1} + 2\, p_{k-1}^n)] \qquad (22)$$

In comparing this program to the non-integrated form SEP-2, p. 113, we find that the differential form is slightly cheaper.

In geophysical calculations we often have a trade-off between memory required and computation required. In reflection seismic data analysis we often choose the route of more calculation in less memory. In the present case the wave field is a function of two space variables and one time variable. To economize on memory we find it extremely expeditious to march along with parallel rows in the z-t plane rather than store the entire plane. This unfortunately results in the necessity to move data around in a frequent, sometimes confusing fashion. The diagram in Figure 8 is intended to clarify the moving. A further complication is that in practice the diffractions really need not be done in every z - step.

It remains to develop a program for the inverse calculation, i.e., data processing. This will naturally be a synthesis of Riley's work (SEP-3) and the slant wave ideas presented here and in SEP-5.

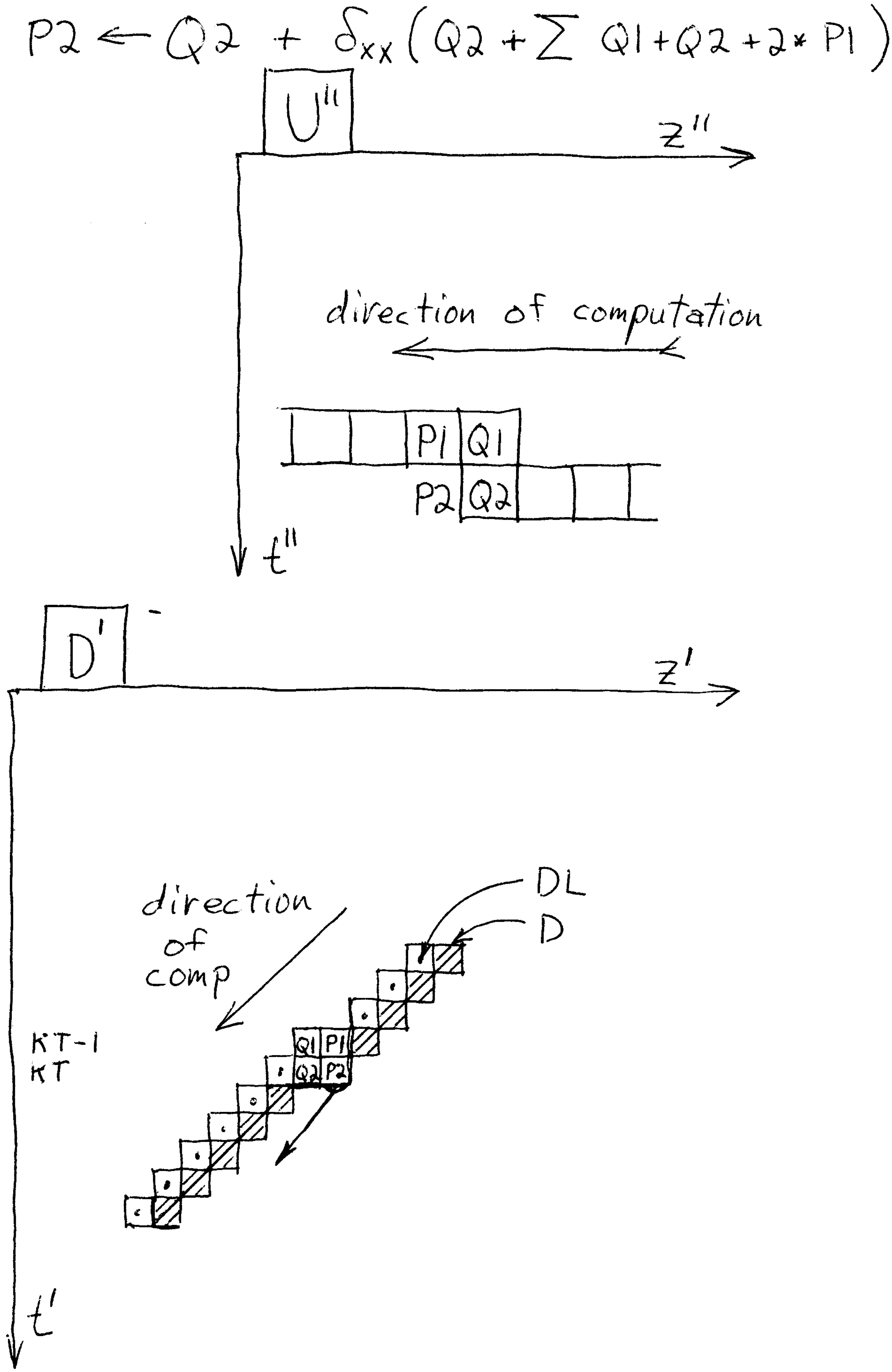


Fig. 8. Ordering of diffraction calculations.