Appendix B

Manipulations of the Multichannel Equations

Considerable effort by the author has been done in attempting to understand and simplify the multichannel equations. The following matrix manipulations have not led to any practical results, but they are put into this thesis as a starting point for future study.

We shall assume that the Nth order Toeplitz matrix in (III-11) is positive definite. Thus, P_{N-1} and P'_{N-1} are positive definite, M by M , Hermitian matrices and can be diagonalized by orthonormal transformations. That is,

$$L^{\dagger}_{N-1} P_{N-1} L_{N-1} = D_{N-1} \quad \text{and} \quad L'^{\dagger}_{N-1} P'_{N-1} L'_{N-1} = D'_{N-1} , \tag{B-1}$$

where $L^{-1}_{N-1} = L^{\dagger}_{N-1}$, $L'^{-1}_{N-1} = L'^{\dagger}_{N-1}$ and D_{N-1} and D'_{N-1} are diagonal matrices containing the necessarily positive eigenvalues of P_{N-1} and P'_{N-1} . Let us define Γ_{N-1} to be a diagonal matrix whose elements are the positive square roots of the reciprocal eigenvalues of D_{N-1} so that $\Gamma_{N-1} \Gamma_{N-1} = D^{-1}_{N-1}$. Likewise, define Γ'_{N-1} so that $\Gamma'_{N-1} \Gamma'_{N-1} = D'^{-1}_{N-1}$. Using Γ_{N-1} and Γ'_{N-1} , we note that

$$\Gamma_{N-1} L^{\dagger}_{N-1} P_{N-1} L_{N-1} \Gamma_{N-1} = \Gamma_{N-1} D_{N-1} \Gamma_{N-1} = I ,$$

and

$$\Gamma'_{N-1} L'^{\dagger}_{N-1} P'_{N-1} L'_{N-1} \Gamma'_{N-1} = \Gamma'_{N-1} D'_{N-1} \Gamma'_{N-1} = I . \tag{B-2}$$

Letting [R] be the Nth order Toeplitz matrix in (III-11), we can formulate an equation similar to (III-11) as

$$
[R]\left[\begin{bmatrix} L_{N-1}\Gamma_{N-1} \\ F_1 L_{N-1}\Gamma_{N-1} \\ \vdots \\ F_{N-1}L_{N-1}\Gamma_{N-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{N-1}L'_{N-1}\Gamma'_{N-1} \\ \vdots \\ B_1 L'_{N-1}\Gamma'_{N-1} \\ L'_{N-1}\Gamma'_{N-1} \end{bmatrix} G_N\right] = \left[\begin{bmatrix} P_{N-1}L_{N-1}\Gamma_{N-1} \\ 0 \\ \vdots \\ 0 \\ \Delta_N L_{N-1}\Gamma_{N-1} \end{bmatrix} + \begin{bmatrix} \Delta_N^{\dagger} L'_{N-1}\Gamma'_{N-1} \\ 0 \\ \vdots \\ 0 \\ P'_{N-1}L'_{N-1}\Gamma'_{N-1} \end{bmatrix} G_N\right] \tag{B-3}
$$

We turn (B-3) into a modified forward prediction error filter equation by letting

$$
\Delta_N \, L_{N-1} \, \Gamma_{N-1} + P'_{N-1} \, L'_{N-1} \, \Gamma'_{N-1} \, G_N = 0 \ . \tag{B-4}
$$

Premultiplying by $\Gamma'_{N-1} \, L'^{\dagger}_{N-1}$ and using (B-2), we get the implied definition of G_N as

$$
G_N \equiv - \Gamma'_{N-1} \, L'^{\dagger}_{N-1} \, \Delta_N \, L_{N-1} \, \Gamma_{N-1} \ . \tag{B-5}
$$

The equation relating G_N with C_N is found by postmultiplying (B-3) by $\Gamma^{-1}_{N-1} \, L^{\dagger}_{N-1}$ and comparing with (III-11) to get

$$
C_N = L'_{N-1} \, \Gamma'_{N-1} \, G_N \, \Gamma^{-1}_{N-1} \, L^{\dagger}_{N-1} \ . \tag{B-6}
$$

The correspondingly modified backward prediction error filter equation is

$$
[R]\left[\begin{bmatrix} L_{N-1}\Gamma_{N-1} \\ F_1 L_{N-1}\Gamma_{N-1} \\ \vdots \\ F_{N-1}L_{N-1}\Gamma_{N-1} \\ 0 \end{bmatrix} G'_N + \begin{bmatrix} 0 \\ B_{N-1}L'_{N-1}\Gamma'_{N-1} \\ \vdots \\ B_1 L'_{N-1}\Gamma'_{N-1} \\ L'_{N-1}\Gamma'_{N-1} \end{bmatrix}\right] = \begin{bmatrix} P_{N-1}L_{N-1}\Gamma_{N-1} \\ 0 \\ \vdots \\ 0 \\ \Delta_N L_{N-1}\Gamma_{N-1} \end{bmatrix} G'_N + \begin{bmatrix} \Delta_N^{\dagger} L'_{N-1}\Gamma'_{N-1} \\ 0 \\ \vdots \\ 0 \\ P'_{N-1}L'_{N-1}\Gamma'_{N-1} \end{bmatrix} . \tag{B-7}
$$

Letting

$$P_{N-1} L_{N-1} \Gamma_{N-1} G'_N + \Delta^{\dagger}_N L'_{N-1} \Gamma'_{N-1} = 0 \quad ,$$

premultiplying by $\Gamma_{N-1} L^{\dagger}_{N-1}$ and using (B-2), we discover that

$$G'_N \equiv - \Gamma_{N-1} L^{\dagger}_{N-1} \Delta^{\dagger}_N L'_{N-1} \Gamma'_{N-1} = G^{\dagger}_N \quad . \tag{B-8}$$

The equation relating G_N with C'_N is found by postmultiplying (B-7) by $\Gamma'^{-1}_{N-1} L'^{\dagger}_{N-1}$ and comparing with (III-11) to get

$$C'_N = L_{N-1} \Gamma_{N-1} G^{\dagger}_N \Gamma'^{-1}_{N-1} L'^{\dagger}_{N-1} \quad . \tag{B-9}$$

Looking at (III-24) and using (B-6) and (B-9), we see that

$$P_N = P_{N-1} [I - C'_N C_N] =$$

$$P_{N-1}[I - L_{N-1} \Gamma_{N-1} G^{\dagger}_N \Gamma'^{-1}_{N-1} L'^{\dagger}_{N-1} L'_{N-1} \Gamma'_{N-1} G_N \Gamma^{-1}_{N-1} L^{\dagger}_{N-1}$$

$$= P_{N-1}[I - L_{N-1} \Gamma_{N-1} G^{\dagger}_N G_N \Gamma^{-1}_{N-1} L^{\dagger}_{N-1}] \tag{B-10}$$

$$= P_{N-1} L_{N-1} \Gamma_{N-1} [I - G^{\dagger}_N G_N] \Gamma^{-1}_{N-1} L^{\dagger}_{N-1}$$

$$= L_{N-1} \Gamma^{-1}_{N-1} [I - G^{\dagger}_N G_N] \Gamma^{-1}_{N-1} L^{\dagger}_{N-1} \quad ,$$

using (B-2) in the last step. The corresponding equation for P'_N is

$$P'_N = L'_{N-1} \Gamma'^{-1}_{N-1} [I - G_N G^{\dagger}_N] \Gamma'^{-1}_{N-1} L'^{\dagger}_{N-1} \quad . \tag{B-11}$$

Let us express G_N as

$$G_N = U_N^{\dagger} S_N V_N \quad , \tag{B-12}$$

where U_N and V_N are orthonormal matrices and S_N is diagonal. This form is always possible for any square matrix and is discussed in (ref 10). The elements of S_N are non-negative, real numbers and are called the singular values of G_N .

Using (B-12), we see that (B-10) becomes

$$\begin{aligned} P_N &= L_{N-1} \Gamma_{N-1}^{-1} V_N^{\dagger} [I - S_N^2] V_N \Gamma_{N-1}^{-1} L_{N-1}^{\dagger} \\ &= [V_N \Gamma_{N-1}^{-1} L_{N-1}^{\dagger}]^{\dagger} [I - S_N^2] [V_N \Gamma_{N-1}^{-1} L_{N-1}^{\dagger}] \quad . \end{aligned} \tag{B-13}$$

From the definition of positive definiteness, we see that P_N will be positive definite if, and only if, $I - S_N^2$ is positive definite or if all of the singular values of G_N are less than unity. Another equivalent statement is that the eigenvalues of $G_N^{\dagger} G_N$ (or $G_N G_N^{\dagger}$) be less than unity. Finally, from (B-6) and (B-9), we have

$$\begin{aligned} C_N' C_N &= L_{N-1} \Gamma_{N-1} G_N^{\dagger} \Gamma_{N-1}'^{-1} L_{N-1}'^{\dagger} L_{N-1}' \Gamma_{N-1}' G_N \Gamma_{N-1}^{-1} L_{N-1}^{\dagger} \\ &= L_{N-1} \Gamma_{N-1} G_N^{\dagger} G_N \Gamma_{N-1}^{-1} L_{N-1}^{\dagger} \\ &= [L_{N-1} \Gamma_{N-1}] G_N^{\dagger} G_N [L_{N-1} \Gamma_{N-1}]^{-1} \quad . \end{aligned}$$

Thus $C_N' C_N$ is a similarity transformation of $G_N^{\dagger} G_N$ and they must have the same eigenvalues. Therefore, our positive definite condition becomes that the eigenvalues of $C_N' C_N$ be less than unity.