Appendix B

Manipulations of the Multichannel Equations

Considerable effort by the author has been done in attempting to understand and simplify the multichannel equations. The following matrix manipulations have not led to any practical results, but they are put into this thesis as a starting point for future study.

We shall assume that the \( N \)th order Toeplitz matrix in (III-11) is positive definite. Thus, \( P_{N-1} \) and \( P'_{N-1} \) are positive definite, \( M \) by \( M \), Hermitian matrices and can be diagonalized by orthonormal transformations. That is,

\[
L_{N-1}^* L_{N-1} = D_{N-1} \quad \text{and} \quad L_{N-1}^* P_{N-1} L_{N-1} = D'_{N-1} \quad ,
\]

where \( L_{N-1}^{-1} = L_{N-1}^* \), \( L'_{N-1}^{-1} = L'_{N-1}^* \) and \( D_{N-1} \) and \( D'_{N-1} \) are diagonal matrices containing the necessarily positive eigenvalues of \( P_{N-1} \) and \( P'_{N-1} \). Let us define \( \Gamma_{N-1} \) to be a diagonal matrix whose elements are the positive square roots of the reciprocal eigenvalues of \( D_{N-1} \) so that \( \Gamma_{N-1}^{-1} \Gamma_{N-1} = D_{N-1}^{-1} \). Likewise, define \( \Gamma'_{N-1} \) so that \( \Gamma'_{N-1}^{-1} \Gamma'_{N-1} = D'_{N-1}^{-1} \). Using \( \Gamma_{N-1} \) and \( \Gamma'_{N-1} \), we note that

\[
\Gamma_{N-1} L_{N-1}^* L_{N-1} \Gamma_{N-1} = \Gamma_{N-1} D_{N-1} \Gamma_{N-1} = \mathbf{I} \quad ,
\]

and

\[
\Gamma'_{N-1} L_{N-1}^* L_{N-1} \Gamma'_{N-1} = \Gamma'_{N-1} D'_{N-1} \Gamma'_{N-1} = \mathbf{I} \quad .
\]

Letting \( [R] \) be the \( N \)th order Toeplitz matrix in (III-11), we can formulate an equation similar to (III-11) as
We turn (B-3) into a modified forward prediction error filter equation by letting

\[
\Delta_N^L L_{N-1}^\Gamma N-1 + P_{N-1}^\Gamma L_{N-1}^\Gamma N-1 G_N = 0. \tag{B-4}
\]

Premultiplying by \( L_{N-1}^\Gamma N-1 \) and using (B-2), we get the implied definition of \( G_N \) as

\[
G_N \equiv - L_{N-1}^\Gamma N-1 \Delta_N^L L_{N-1}^\Gamma N-1. \tag{B-5}
\]

The equation relating \( G_N \) with \( C_N \) is found by postmultiplying (B-3) by \( L_{N-1}^{-1} L_{N-1}^\Gamma \) and comparing with (III-11) to get

\[
C_N = L_{N-1}^{-1} L_{N-1}^\Gamma G_N \Delta_N^L L_{N-1}^{-1} L_{N-1}^\Gamma. \tag{B-6}
\]

The correspondingly modified backward prediction error filter equation is

\[
\begin{bmatrix}
L_{N-1}^\Gamma N-1 \\
F_{N-1}^\Gamma N-1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
B_{N-1}^\Gamma N-1 \\
L_{N-1}^\Gamma N-1
\end{bmatrix}
G_N + \begin{bmatrix}
P_{N-1}^\Gamma N-1 \\
B_{N-1}^\Gamma N-1 \\
L_{N-1}^\Gamma N-1
\end{bmatrix}
\begin{bmatrix}
\Delta_{N-1}^L N-1 \\
\Delta_{N-1}^L N-1 \\
\Delta_{N-1}^L N-1
\end{bmatrix}
\begin{bmatrix}
G_N \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
P_{N-1}^\Gamma N-1 \\
B_{N-1}^\Gamma N-1 \\
L_{N-1}^\Gamma N-1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\Delta_{N-1}^L N-1 \\
\Delta_{N-1}^L N-1 \\
\Delta_{N-1}^L N-1
\end{bmatrix}
\begin{bmatrix}
G_N \\
0 \\
0
\end{bmatrix}. \tag{B-7}
\]
Letting

\[ p_{N-1} L_{N-1} \Gamma_{N-1} C_N' + \Delta_N^+ L_{N-1} \Gamma_{N-1}' = 0, \]

premultiplying by \( \Gamma_{N-1}' L_{N-1}^- \) and using (B-2), we discover that

\[ G_N' = -\Gamma_{N-1}' L_{N-1}^- \Delta_N^+ L_{N-1}^- \Gamma_{N-1} = C_N^+. \tag{B-8} \]

The equation relating \( G_N' \) with \( C_N' \) is found by postmultiplying (B-7) by \( \Gamma_{N-1}' L_{N-1}^+ \) and comparing with (III-11) to get

\[ C_N' = L_{N-1}^- \Gamma_{N-1} C_N^+ \Gamma_{N-1}' L_{N-1}^- \Gamma_{N-1} \tag{B-9} \]

Looking at (III-24) and using (B-6) and (B-9), we see that

\[ p_N = p_{N-1} \left[ I - C_N' C_N \right] = \]

\[ p_{N-1} \left[ I - L_{N-1} \Gamma_{N-1} G_N^+ \Gamma_{N-1}' L_{N-1}^+ \Gamma_{N-1}^- L_{N-1}^+ \Gamma_{N-1}^- L_{N-1}^+ \right] \]

\[ = p_{N-1} \left[ I - L_{N-1} \Gamma_{N-1} G_N^+ G_N \Gamma_{N-1}' L_{N-1}^+ \right] \]

\[ = p_{N-1} L_{N-1} \Gamma_{N-1} \left[ I - G_N^+ G_N \right] \Gamma_{N-1}' L_{N-1}^+ \]

\[ = L_{N-1} \Gamma_{N-1} \left[ I - G_N^+ G_N \right] \Gamma_{N-1}' L_{N-1}^+, \tag{B-10} \]

using (B-2) in the last step. The corresponding equation for \( p_N' \) is

\[ p_N' = L_{N-1} \Gamma_{N-1}' \left[ I - C_N' C_N \right] \Gamma_{N-1}^- L_{N-1}^+ \tag{B-11} \]
Let us express $G_N$ as

$$G_N = U_N^T S_N V_N,$$  \hspace{1cm} (B-12)

where $U_N$ and $V_N$ are orthonormal matrices and $S_N$ is diagonal. This form is always possible for any square matrix and is discussed in (ref 10). The elements of $S_N$ are non-negative, real numbers and are called the singular values of $G_N$.

Using (B-12), we see that (B-10) becomes

$$P_N = L_{N-1} \Gamma_{N-1}^{-1} V_N^T \left[ I - S_N^2 \right] V_N \Gamma_{N-1}^{-1} L_{N-1}^T$$

$$= \left[ V_N \Gamma_{N-1}^{-1} L_{N-1}^T \right]^T \left[ I - S_N^2 \right] \left[ V_N \Gamma_{N-1}^{-1} L_{N-1}^T \right].$$ \hspace{1cm} (B-13)

From the definition of positive definiteness, we see that $P_N$ will be positive definite if, and only if, $I - S_N^2$ is positive definite or if all of the singular values of $G_N$ are less than unity. Another equivalent statement is that the eigenvalues of $G_N^+ C_N$ (or $G_N C_N^+$) be less than unity. Finally, from (B-6) and (B-9), we have

$$C_N^T C_N = L_{N-1} \Gamma_{N-1} \Gamma_{N-1}^{-1} L_{N-1}^T \Gamma_{N-1}^T \Gamma_{N-1}^T L_{N-1}^T$$

$$= L_{N-1} \Gamma_{N-1} \Gamma_{N-1}^{-1} \Gamma_{N-1} \Gamma_{N-1}^{-1} L_{N-1}^T$$

$$= \left[ I_{N-1} \Gamma_{N-1} \right] G_N^+ G_N \left[ I_{N-1} \Gamma_{N-1} \right]^{-1}.$$

Thus $C_N^T C_N$ is a similarity transformation of $G_N^+ G_N$ and they must have the same eigenvalues. Therefore, our positive definite condition becomes that the eigenvalues of $C_N^T C_N$ be less than unity.