Appendix A

Determination of Pure Line Spectra

From expression (II-44) in II-B.7, we see that an Nth order maximum entropy spectrum, $P_N(f)$, can be written as

$$P_N(f) = \frac{R(0)\Delta t}{2} \frac{D^*(z^{-1}) - c_N^* z^{-N} D(z)}{A^*(z^{-1}) + c_N^* z^{-N} A(z)} + \frac{R(0)\Delta t}{2} \frac{D(z) - c_N z^N D^*(z^{-1})}{A(z) + c_N z^N A^*(z^{-1})} , \qquad \text{(A-1)}$$

where $A(z)$ is the N-1th order prediction error filter and $D(z)$ is the N-1th order prediction error filter corresponding to the negative sequence of reflection coefficients. Let us factor the Nth order prediction error filter, i.e.,

$$A(z) + c_N z^N A^*(z^{-1}) = \prod_{n=1}^{N} (1 - \beta_n z) , \qquad \text{(A-2)}$$

where β_n^{-1} (n=1 to N) are the N roots of the polynomial.

By doing a partial fraction expansion, we can write

$$\frac{R(0)\Delta t}{2} \frac{D(z) - c_N z^N D^*(z^{-1})}{A(z) + c_N z^N A^*(z^{-1})} = \sum_{n=1}^{N} \frac{B_n \Delta t}{1-\beta_n z} - \frac{R(0)\Delta t}{2} . \qquad \text{(A-3)}$$

The constant term on the right-hand side occurs because the numerator and denominator polynomials on the left-hand side are both of order N and their highest order coefficients are $-c_N$ and c_N respectively. By letting z go to infinity, we confirm that the value of the constant term is $- R(0)\, \Delta t / 2$. Furthermore, since the numerator and denominator polynomials on the left-hand side both start with unity, we discover by setting z to zero that

$$\sum_{n=1}^{N} B_n \Delta t = R(0) \Delta t .$$

Using this, we can write

$$\frac{R(0)\Delta t}{2} \frac{D(z) - c_N z^N D^*(z^{-1})}{A(z) + c_N z^N A^*(z^{-1})} = \frac{\Delta t}{2} \sum_{n=1}^{N} B_n \frac{1+\beta_n z}{1-\beta_n z} .$$

The value of B_n may be found by multiplying by $1-\beta_n z$ and setting $z = \beta_n^{-1}$.

From (A-1), we can write

$$P_N(f) = \frac{\Delta t}{2} \sum_{n=1}^{N} \left[B_n^* \frac{1+\beta_n^* z^{-1}}{1-\beta_n^* z^{-1}} + B_n \frac{1+\beta_n z}{1-\beta_n z} \right] . \qquad \text{(A-4)}$$

This expression gives the maximum entropy spectrum as a sum of poles instead of a product of poles. We shall now look at the properties of the individual summation terms.

We should first note that for $|z|=1$, the expression

$$\frac{\Delta t}{2} \left[B^* \frac{1+\beta^* z^{-1}}{1-\beta^* z^{-1}} + B \frac{1+\beta z}{1-\beta z} \right] \qquad \text{(A-5)}$$

is real. Also, the total area under this curve is equal to the real part of B .

To prove this, let us consider the function

$$\frac{1}{4\pi i} \ln \left[(1-\beta z)^2 z^{-1} \right] ,$$

whose real part is the phase of $(1-\beta z)^2 z^{-1}$ divided by 4π . With $z = e^{-i2\pi f \Delta t}$ and $dz = -i2\pi\Delta t\, z\, df$, the derivative with respect to f is

$$\frac{d}{df} \left\{ \frac{1}{4\pi i} \ln \left[(1-\beta z)^2 z^{-1} \right] \right\} = \frac{-\Delta t z}{2} \frac{d}{dz} \left[\ln (1-\beta z)^2 z^{-1} \right]$$

$$= \frac{\Delta t\, z}{2} \left[\frac{1}{z} + \frac{2\beta}{1-\beta z} \right] = \frac{\Delta t}{2} \left[1 + \frac{2\beta z}{1-\beta z} \right] = \frac{\Delta t}{2} \left(\frac{1+\beta z}{1-\beta z} \right) . \qquad \text{(A-6)}$$

Using this equation and its complex conjugate, we thus have that

$$\int_{-W}^{f} \frac{\Delta t}{2} \left[B^* \frac{1+\beta^* z^{-1}}{1-\beta^* z^{-1}} + B \frac{1+\beta z}{1-\beta z} \right] df =$$

$$= \left\{ \frac{B^*}{4\pi i} \ln \left[(1-\beta^* z^{-1})^2 z \right] + \frac{B}{4\pi i} \ln \left[(1-\beta z)^2 z^{-1} \right] \right\} \Bigg|_{-W}^{f} . \qquad \text{(A-7)}$$

To evaluate (A-7) when $f=W$, we note that each of the logarithmic functions has a net phase shift of 2π, remembering that $(1-\beta z)$ has no net phase shift but $(1-\beta^* z^{-1})$ does. Thus, the total integral is $\frac{1}{2}(B^*+B)$ = Real Part of B.

We can also note that by putting (A-7) into the integral of (A-4) with respect to f from $-W$ to f, we have an analytic form for the integrated power spectrum of a maximum entropy spectrum.

In looking at the functional form of (A-5), we see that if B is real and positive, then (A-5) is a first order maximum entropy spectrum with $B = R(0)$ and $c_1 = -\beta$. However, in general B will not be real and thus the simple interpretation of (A-4) as the sum of N first order maximum entropy spectra is not valid. For a more detailed look at (A-5), let β be real. This simply rotates the function so that the two poles are on the real axis. Then

$$\frac{\Delta t}{2} \left[B^* \frac{1+\beta z^{-1}}{1-\beta z^{-1}} + B \frac{1+\beta z}{1-\beta z} \right] =$$

$$\frac{\Delta t}{2} \left[\frac{(B^*-B)\beta z^{-1} + (B+B^*)(1-\beta^2) + (B-B^*)\beta z}{(1-\beta z^{-1})(1-\beta z)} \right] =$$

$$\Delta t \left[\frac{B_R(1-\beta^2) + 2\beta B_I \sin(2\pi f \Delta t)}{1+\beta^2 - 2\beta\cos(2\pi f \Delta t)} \right] , \qquad \text{(A-8)}$$

where B_R and B_I are the real and imaginary parts of B .

As we have just noted, if $B_I=0$, then we would have a first order maximum entropy spectrum. However, the imaginary part of B skews the function and, indeed, if $|2\beta B_I| > |B_R(1-\beta^2)|$, then (A-8) will actually be negative for some values of f . Because of this, (A-4) is not a particularly useful form for studying maximum entropy spectra in general. However we shall use (A-4) in the following study of the special situation in which $|c_N|$ goes to unity so that the spectrum turns into a set of N spectral lines.

As shown in II-B.8, when $|c_N|=1$, the zeros of the prediction error filter all lie on the unit circle and are distinct. Consequently, the spectrum consists of a set of N delta functions. However, in this limiting case, the mean square error is zero and the strength of the delta functions cannot be determined from the prediction error filter. We shall develop here an expression for the strength of the delta functions in terms of the N-1th prediction error filter. This will lead to a new positive real functional form involving a prediction error filter and to a better understanding of spectra consisting of pure spectral lines.

Multiplying (A-4) through by $(1-\beta_n z)$, setting $z = \beta_n^{-1}$ and using (II-45), we see that

$$B_n = \frac{1}{\Delta t} P(f)(1-\beta_n z)\Big|_{z=\beta_n^{-1}}$$

$$= \frac{R(0)\prod_{m=1}^{N}(1-|c_m|^2)}{A^*(z^{-1})+c_N^* z^{-N} A(z)} \quad \frac{1-\beta_n z}{A(z)+c_N z^N A^*(z^{-1})}\Bigg|_{z=\beta_n^{-1}}$$

or by using l'Hospital's rule,

$$= \frac{R(0) \prod_{m=1}^{N} (1 - |c_m|^2)}{A^*(z^{-1}) + c_N^* z^{-N} A(z)} \quad \frac{-\beta_n}{[\,A(z) + c_N z^N A^*(z^{-1})\,]' \,\Big|_{z=\beta_n^{-1}}} , \qquad \text{(A-9)}$$

where the prime indicates the derivative with respect to z. This derivative is given by

$$A'(z) + N\, c_N\, z^{N-1}\, A^*(z^{-1}) + c_N\, z^N\, A^{*\prime}(z^{-1})(-z^{-2}) \quad , \qquad \text{(A-10)}$$

where $A^{*\prime}(\ )$ means the derivative of the function $A^*(\ )$. Since $z = \beta_n^{-1}$ is a root of $A(z) + c_N z^N A^*(z^{-1})$ and we wish to evaluate (A-10) at $z = \beta_n^{-1}$, we can use the equation $c_N z^N A^*(z^{-1}) = -A(z)\big|_{z=\beta_n^{-1}}$ to eliminate the explicit dependence of (A-10) on c_N. Thus, when $z = \beta_n^{-1}$, (A-10) becomes

$$A'(z) - N\, z^{-1}\, A(z) + z^{-2}\, A(z)\, A^{*\prime}(z^{-1}) \,/\, A^*(z^{-1}) \; \Big|_{z=\beta_n^{-1}} \quad . \qquad \text{(A-11)}$$

Putting (A-11) into (A-9), would give us a general expression for B_n.

At this point, we shall let $|c_N|$ go to unity. $P_N(f)$ then becomes a set of N delta functions located at frequencies, f_n, where β_n^{-1} are the N unit magnitude roots of $A(z) + c_N z^N A^*(z^{-1})$ and $e^{-i2\pi\Delta t f_n} = \beta_n^{-1}$. The strength of the nth delta function is given by the real part of B_n.

When $|c_N|=1$, the first factor in (A-9) is an indeterminant form at $z=\beta_n^{-1}$ since this value of z will also be a root of $A^*(z^{-1}) + c_N^* z^{-N} A(z)$. To resolve this indeterminancy, let us replace c_N by $\alpha\, c_N$, where α is positive and slightly less than unity. Then, for $z=\beta_n^{-1}$, we can write the first factor of (A-9) as

$$\frac{R(0)(1-\alpha^2)\prod_{m=1}^{N-1}(1-|c_m|^2)}{A^*(z^{-1})+\alpha c_N^* z^{-N} A(z)} = \frac{R(0)(1+\alpha)(1-\alpha)\prod_{m=1}^{N-1}(1-|c_n|^2)}{A^*(z^{-1})(1-\alpha)} .$$

Letting α go to unity, we then get

$$\left.\frac{2\ R(0)\prod_{m=1}^{N-1}(1-|c_m|^2)}{A^*(z^{-1})}\right|_{z=\beta_n^{-1}} . \tag{A-12}$$

Combining (A-9), (A-11) and (A-12), we get

$$B_n = \left.\frac{2\ R(0)\prod_{m=1}^{N-1}(1-|c_m|^2)\qquad(-\beta_n)}{A^*(z^{-1})[A'(z)-N z^{-1}A(z)+z^{-2}A(z)A^{*'}(z^{-1})/A^*(z^{-1})]}\right|_{z=\beta_n^{-1}}$$

$$= \left.\frac{2R(0)\prod_{m=1}^{N-1}(1-|c_m|^2)}{[N\ A(z)A^*(z^{-1})-z\ A'(z)A^*(z^{-1})-z^{-1}\ A(z)A^{*'}(z^{-1})]}\right|_{z=\beta_n^{-1}} . \tag{A-13}$$

Since $|\beta_n|=1$, we have $\beta_n^{-1}=\beta_n^*$. Thus, for $z=\beta_n^{-1}$, $A^{*'}(z^{-1}) = A^{*'}(z^*) = [A'(z)]^*$. Using this, and factoring out $A(z)\ A^*(z^{-1})$, we obtain from (A-13)

$$B_n = \frac{2\ R(0)\prod_{m=1}^{N-1}(1-|c_m|^2)}{A(z)\ A^*(z^{-1})}\ \left.\frac{1}{\{N-z\frac{A'(z)}{A(z)}-[z\frac{A'(z)}{A(z)}]^*\}}\right|_{z=\beta_n^{-1}} . \tag{A-14}$$

Equation (A-14) is most interesting. We can first note that the first factor is two times the N-1th order maximum entropy spectrum. The second factor is clearly real and thus B_n is real. B_n is the strength of the delta function at f_n and since the corresponding autocorrelation matrix is non-negative definite, B_n must be positive.

Looking at (A-14) again, we note that c_N does not explicitly appear in the expression but that the value of B_n is determined by the location of the delta function on the unit circle. Since we have seen in II-B.8 that a delta function can be located anywhere on the unit circle by properly adjusting the phase of c_N, we shall drop the subscript n and consider (A-14) as a z-transform expression. In particular, we shall study the properties of the denominator of the second factor where it is written in the analytic form of

$$N - z\,\frac{A'(z)}{A(z)} - z^{-1}\,\frac{A^{*'}(z^{-1})}{A^{*}(z^{-1})} \quad . \tag{A-15}$$

Since B is positive for all $|z| = 1$, (A-15) is an autocorrelation function. From this, it is clear that

$$\frac{N}{2} - z\,\frac{A'(z)}{A(z)} \tag{A-16}$$

is a positive real function.

From II-B.7, we saw that the delta functions were located at the roots of $A(z) + c_N\, z^N A^*(z^{-1}) = 0$. An equivalent statement is that the phase of

$$\frac{-\,c_N\, z^N A^*(z^{-1})}{A(z)} \tag{A-17}$$

is zero (or an integral multiple of 2π), or that

$$\phi = \frac{1}{i}\,\ell n\,\frac{-\,c_N\, z^N A^*(z^{-1})}{A(z)} = n\,2\pi \quad . \tag{A-18}$$

Let $z = e^{i\omega}$ so that $dz = i\,z\,d\omega$. Then

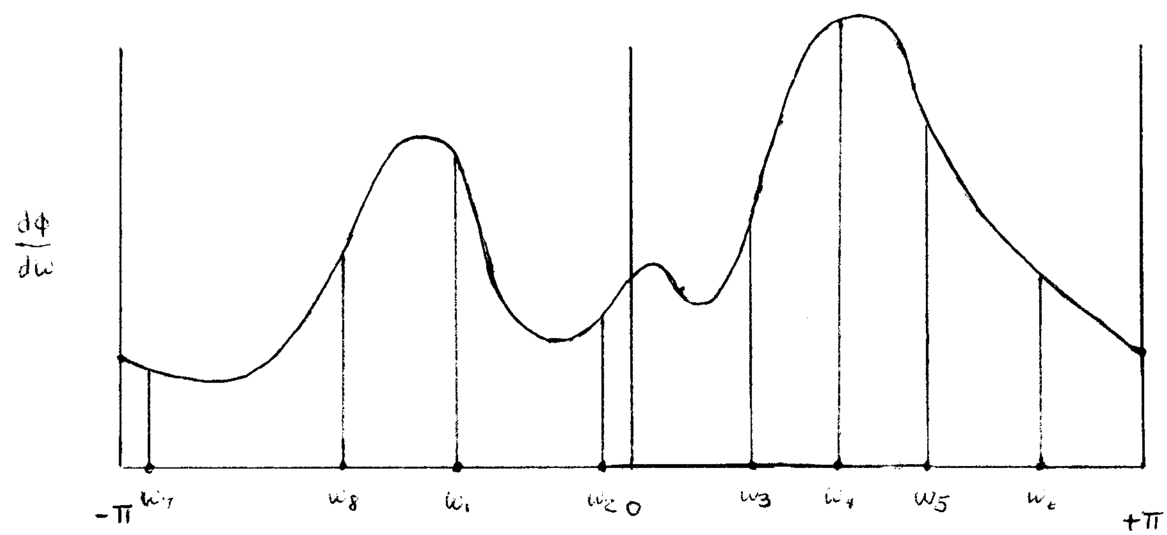
$$\frac{d\phi}{d\omega} = iz\frac{d\phi}{dz} = z\frac{d}{dz}\ln\frac{-c_N z^N A^*(z^{-1})}{A(z)} =$$

$$z\left[\frac{N}{z} + \frac{A^{*\prime}(z^{-1})(-z^{-2})}{A^*(z^{-1})} - \frac{A'(z)}{A(z)}\right]$$

$$= N - z\frac{A'(z)}{A(z)} - z^{-1}\frac{A^{*\prime}(z^{-1})}{A^*(z^{-1})} . \tag{A-19}$$

Thus, (A-15) is the rate of change of phase of (A-17) with respect to ω . From inspection, as ω goes around once, the phase of (A-17) makes N circuits. Thus the integral of (A-15) with respect to ω from $-\pi$ to π is $N\,2\pi$.

Suppose we graph (A-15) with respect to ω for $N=8$ and assume that the phase of c_N is such that one of the delta functions is at ω_1 .



Then the other delta functions are located at points such that the area under the curve between the points is 2π . The strength of the delta function at ω_n is then found as two times the value of the N-1th maximum entropy spectrum at ω_n divided by the value of $d\phi/d\omega$ at ω_n .

We note that as the phase of c_N increases by 2π , the locations of the delta functions move to the left until ω_n becomes ω_{n-1} .

We know that $d\phi/d\omega$ is positive since (A-17) is the z-transform of a physically realizable all-pass filter (see Claerbout, 1976)*. Actually, this is still true for the same reason if z^N is replaced by z^{N-1} in (A-17) or if N is replaced by $N-1$ in (A-15). Thus, a sharper form of (A-16) is that

$$\frac{N}{2} - z\,\frac{A'(z)}{A(z)} \tag{A-20}$$

is positive real where $A(z)$ is an Nth order prediction error filter instead of being $N-1$th order. A geometrical proof of this, using $z = e^{i\omega}$, follows.

The phase, ϕ, of $A(z)$ can be written as

$$\phi = \text{Real Part of } \frac{1}{i} \ln A(z) \;,$$

and its derivative with respect to ω is

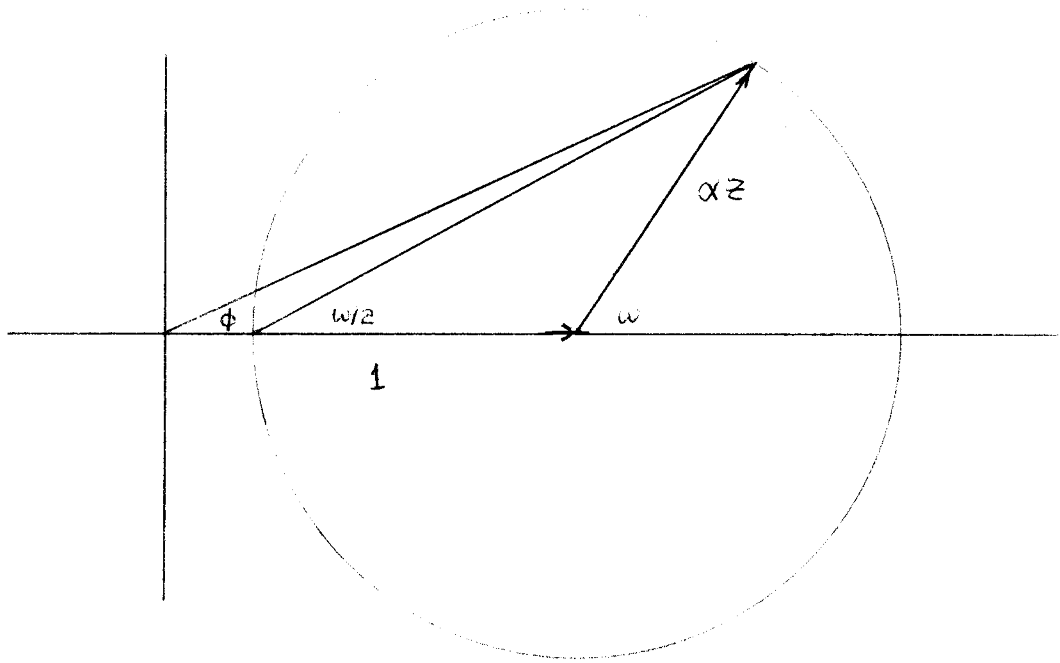
$$\frac{d\phi}{d\omega} = \text{Real Part of } \frac{1}{i}\,\frac{d\ln A(z)}{d\omega} =$$

$$\text{Real Part of } z\,\frac{d\ln A(z)}{dz} = \text{Real Part of } z\,\frac{A'(z)}{A(z)} \;.$$

Thus, for (A-20) to be positive real, the rate of change of phase of $A(z)$ with respect to ω must be less than $N/2$.

Suppose we look at the phase properties of a first order minimum phase filter, $1 + \alpha z$, where $|\alpha| < 1$ and we take α real and positive for convenience. As z goes around the unit circle, $1 + \alpha z$ has the following locus.

* Claerbout, Jon F., Fundamentals of Geophysical Data Processing, McGraw-Hill, 1976.



From plane geometry, we see that $\Delta\phi < \Delta\omega/2$. We might note that when $\omega \sim \pi$, $|\Delta\phi| > |\Delta\omega|/2$ but that at this time, $d\phi/d\omega$ is negative.

Since there are N such factors in the Nth order filter, we see that

$$\frac{N}{2} - \text{Real Part of } z \frac{A'(z)}{A(z)} > 0$$

and thus (A-20) is positive real.

Actually, if we look at equation (A-6) again, we see that we have already proved the above theorem, since the righthand side is a positive real function, which says that the derivative of the phase of $(1-\beta z)^2 z^{-1}$ with respect to f is positive.