III-B. SOLUTION OF THE MULTICHANNEL VARIATIONAL FORMULATION

The constraint or measurement equations that we shall deal with are

$$\int_{-W}^{+W} P(z)\, e^{i2\pi f n \Delta t}\, df = R(n) \text{ , or}$$

$$\int_{-W}^{+W} P(z)\, z^{n}\, df = R(-n) \text{ , } \quad n = -N \text{ to } +N \text{ ,} \qquad \text{(III-3)}$$

where $R(n)$ will be recognized as the M by M cross-correlation matrix at lag n of the multichannel time series.

Our problem is to find the multichannel spectrum that satisfies (III-3) and maximizes

$$\int_{-W}^{+W} \ln [\det P(\ )]\, df \text{ .} \qquad \text{(III-4)}$$

To do this, we shall use Lagrange multipliers.

1. Deriving the Multichannel Prediction Error Filter Equation

We note that (III-3) consists of $(2N+1)M^2$ equations. We thus need $(2N+1)M^2$ Lagrange multipliers, $\lambda_{ij}(n)$, where i, j specify the matrix element in the nth equation. Our variation is thus taken over all M^2 functions in the spectral matrix $P(z)$ to give

$$\delta \int_{-W}^{W} \{ \ln |P(z)| - \sum_{i,j,n} \lambda_{ij}(n)\, [\, P_{ij}(z)\, z^{n} - R_{ij}(-n)/2W \,] \}\, df = 0 \text{ , or}$$

$$\int_{-W}^{W} \sum_{i,j} \left[\frac{Q_{ij}(z)}{|P(z)|} - \sum_{n=-N}^{+N} \lambda_{ij}(n)\, z^{n} \right] \delta P_{ij}(z)\, df = 0$$

where Q_{ij} is the cofactor of $P_{ij}(z)$ in the $P(z)$ matrix. Thus

$$\frac{Q_{ij}(z)}{|P(z)|} = P^{-1}_{ij}(z) \text{ ,}$$

and we have

$$P^{-1}_{ij}(z) = \sum_{n=-N}^{+N} \lambda_{ij}(n)\, z^n \quad , \quad \text{or}$$

$$P^{-1}(z) = \sum_{n=-N}^{+N} \lambda(n)\, z^n \quad , \qquad \text{(III-5)}$$

where $\lambda(n)$ is the matrix of the nth Lagrange multipliers. We see from (II-6) that the reciprocal of the single channel maximum entropy spectrum has a finite length autocorrelation function. Equation (III-5) expresses the multichannel form of this statement, that is, the inverse of the multichannel maximum entropy spectrum has a finite length multichannel autocorrelation function.

We now assert that if our constraint equations (III-3) are consistent, then it must be possible to write

$$\sum_{n=-N}^{+N} \lambda(n)\, z^n = F(z)\, P_N^{-1}\, F^{\dagger}(z^{-1}) \quad , \qquad \text{(III-6)}$$

where $F(z) = F_0 + F_1 z + F_2 z + \ldots + F_N z^N$, $F_0 \equiv I$, is an Nth order multichannel prediction error filter and $P_N^{-1} = P_N^{-1\dagger}$ is a constant M by M power density matrix. We then have

$$P^{-1}(z) = F(z)\, P_N^{-1}\, F^{\dagger}(z^{-1}) \quad , \quad \text{or}$$

$$F^{-1}(z)\, P^{-1}(z) = P_N^{-1}\, F^{\dagger}(z^{-1}) \quad ,$$

or by taking inverses of each side,

$$P(z)\, F(z) = F^{-1\dagger}(z^{-1})\, P_N \quad .$$

Since $F(z)$ is minimum phase, $F^{-1\dagger}(z^{-1})$ contains no positive powers of z . Thus, the left hand side also cannot contain positive powers of z. Since

$$P(z) = \sum_{-\infty}^{+\infty} R(n)\, z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{N} F_n\, z^n ,$$

equality of the zero th power of z gives

$$\sum_{n=0}^{N} R(-n)\, F_n = P_N$$

and equality of the rth power of z when r is positive gives

$$\sum_{n=0}^{N} R(r-n)\, F_n = 0 \; .$$

These equations can be written as the multichannel prediction error filter matrix equation

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(-N+1) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \end{bmatrix} \; . \qquad \text{(III-7)}$$

We note that the square matrix in (III-7) is made up of our measured cross-correlation values and that if this matrix is positive definite, we obtain a unique solution for F_n and P_N. The maximum entropy solution is then given by

$$P(z) = F^{-1\dagger}(z^{-1})\, P_N\, F^{-1}(z) \; . \qquad \text{(III-8)}$$

Since the direction of time has no fundamental importance in spectral analysis, the maximum entropy spectrum can also be derived in terms of the multichannel backward prediction error filter. This is done by writing (III-6) in the alternative form of

$$\sum_{n=-N}^{+N} \lambda(n)\, z^{n} = B(z^{-1})\, P_N'^{-1}\, B^{\dagger}(z) , \qquad \text{(III-6A)}$$

where $B(z) = B_0 + B_1 z + \ldots + B_N z^N$, $B_0 \equiv I$, is an Nth order multichannel prediction error filter and $P_N'^{-1} = P_N'^{-1\dagger}$ is a constant M by M power density matrix. We then have

$$P^{-1}(z) = B(z^{-1})\, P_N'^{-1}\, B^{\dagger}(z) , \quad \text{or}$$

$$B^{-1}(z^{-1})\, P^{-1}(z) = P_N'^{-1}\, B^{\dagger}(z) ,$$

or by taking inverses of each side,

$$P(z)\, B(z^{-1}) = B^{-1\dagger}(z)\, P_N' .$$

Since $B(z)$ is minimum phase, $B^{-1\dagger}(z)$ contains no negative powers of z . Thus, the left hand side also cannot contain negative powers of z . Since

$$P(z) = \sum_{-\infty}^{+\infty} R(n)\, z^{n} \quad \text{and} \quad B(z) = \sum_{n=0}^{N} B_n\, z^{n} ,$$

equality of the zero th power of z gives

$$\sum_{n=0}^{N} R(n)\, B_n = P_N$$

and equality of the rth power of z when r is negative gives

$$\sum_{n=0}^{N} R(n+r)\, B_n = 0 \; .$$

These equations can be written as the backward multichannel prediction error filter matrix equation

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{bmatrix} B_N \\ B_{N-1} \\ \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ P'_N \end{bmatrix} \; . \qquad \text{(III-7A)}$$

The alternative form of the maximum entropy solution is then given by

$$P(z) = B^{-1\dagger}(z)\, P'_N\, B^{-1}(z^{-1}) \; . \qquad \text{(III-8A)}$$

We shall now show how (III-7) and (III-7A) can be solved efficiently by use of the multichannnel version of the Levinson algorithm.

2. The Multichannel Levinson Algorithm[8]

The extension of the single channel Levinson algorithm to the multichannel case is complicated by the fact that the multichannel backward prediction error filter is not just the complex conjugate time reverse of the multichannel forward prediction error filter. This fact requires that the backward filter be calculated explicitly along with the forward filter at each step of the recursion.

In the following we shall assume that there are M channels and thus the block Toeplitz matrix equation consists of M by M matrices. Assuming that the N by N block Toeplitz matrix is positive definite and that we have the solutions to the N-1th order forward and backward multichannel prediction error filter equations, we shall develop the solution to the Nth order forward and backward equations.

The N-1th order forward prediction error equation is

$$\begin{bmatrix} R(0) & R(-1) & & R(1-N) \\ R(1) & R(0) & & R(2-N) \\ & & & \\ R(N-1) & R(N-2) & & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} P_{N-1} \\ 0 \\ \\ 0 \end{bmatrix} \qquad \text{(III-9)}$$

and the N-1th order backward prediction error filter equation is

$$\begin{bmatrix} R(0) & R(-1) & & R(1-N) \\ R(1) & R(0) & & R(2-N) \\ & & & \\ R(N-1) & R(N-2) & & R(0) \end{bmatrix} \begin{bmatrix} B_{N-1} \\ B_{N-2} \\ \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ P'_{N-1} \end{bmatrix}. \qquad \text{(III-10)}$$

Using these equations, we see that

$$\begin{bmatrix} R(0) & R(-1) & & R(1-N) & R(-N) \\ R(1) & R(0) & & R(2-N) & R(1-N) \\ & & & & \\ R(N-1) & R(N-2) & & R(0) & R(-1) \\ R(N) & R(N-1) & & R(1) & R(0) \end{bmatrix} \left\{ \begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{N-1} \\ \\ B_1 \\ I \end{bmatrix} [C_N] \right\} = \left\{ \begin{bmatrix} P_{N-1} \\ 0 \\ \\ 0 \\ \Delta_N \end{bmatrix} + \begin{bmatrix} \Delta'_N \\ 0 \\ \\ 0 \\ P'_{N-1} \end{bmatrix} [C_N] \right\} , \tag{III-11}$$

where

$$\Delta_N = \sum_{n=0}^{N-1} R(N-n)\, F_n \tag{III-12}$$

and

$$\Delta'_N = \sum_{n=0}^{N-1} R(n-N)\, B_n \; , \tag{III-13}$$

with $F_0 = B_0 \equiv I$. We can immediately prove that $\Delta'_N = \Delta^{\dagger}_N$ by noting the matrix equation

$$[\; 0 \quad B^{\dagger}_{N-1} \quad\quad B^{\dagger}_1 \quad I \;] \begin{bmatrix} R(0) & R(-1) & & R(1-N) & R(-N) \\ R(1) & R(0) & & R(2-N) & R(1-N) \\ & & & & \\ R(N-1) & R(N-2) & & R(0) & R(-1) \\ R(N) & R(N-1) & & R(1) & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \\ 0 \end{bmatrix} = [\Delta_N] \; ,$$

is clearly true from (III-11). If we take the complex conjugate transpose of this equation, the block Toeplitz matrix is unchanged and (III-11) says that the right hand side, which will be $\Delta^{\dagger}_N$, is Δ'_N .

If we define

$$C_N = - P'^{-1}_{N-1} \Delta_N \quad , \tag{III-14}$$

so that

$$\Delta_N + P'_{N-1} C_N = 0 \quad , \tag{III-15}$$

then (III-11) becomes the Nth order forward prediction error filter equation. The Nth order forward filter is thus

$$\begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{N-1} \\ \\ B_1 \\ I \end{bmatrix} [C_N] \quad , \tag{III-16}$$

and the Nth order forward mean square error matrix is

$$P_N = P_{N-1} + \Delta_N^{\dagger} C_N \quad . \tag{III-17}$$

Likewise, if we consider

$$\begin{bmatrix} R(0) & R(-1) & & R(1-N) & R(-N) \\ R(1) & R(0) & & R(2-N) & R(1-N) \\ & & & & \\ R(N-1) & R(N-2) & & R(0) & R(-1) \\ R(N) & R(N-1) & & R(1) & R(0) \end{bmatrix} \left\{ \begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \\ 0 \end{bmatrix} [C'_N] + \begin{bmatrix} 0 \\ B_{N-1} \\ \\ B_1 \\ I \end{bmatrix} \right\} = \left\{ \begin{bmatrix} P_{N-1} \\ 0 \\ \\ 0 \\ \Delta_N \end{bmatrix} [C'_N] + \begin{bmatrix} \Delta_N^{\dagger} \\ 0 \\ \\ 0 \\ P'_{N-1} \end{bmatrix} \right\} ,$$

and define

$$C'_N = - P^{-1}_{N-1} \Delta^{\dagger}_N , \qquad \text{(III-19)}$$

so that

$$\Delta^{\dagger}_N + P_{N-1} C'_N = 0 , \qquad \text{(III-20)}$$

then (III-18) becomes the Nth order backward prediction error filter equation. The Nth order backward filter is thus

$$\begin{bmatrix} I \\ F_1 \\ \\ F_{N-1} \\ 0 \end{bmatrix} [C'_N] + \begin{bmatrix} 0 \\ B_{N-1} \\ \\ B_1 \\ I \end{bmatrix} \qquad \text{(III-21)}$$

and the Nth order backward mean square error matrix is

$$P'_N = P'_{N-1} + \Delta_N C'_N . \qquad \text{(III-22)}$$

If one eliminates Δ_N by use of (III-15) , then we have

$$P'_N = P'_{N-1} - P'_{N-1} C_N C'_N = P'_{N-1}[I - C_N C'_N] . \qquad \text{(III-23)}$$

Likewise, putting (III-20) into (III-17), we get

$$P_N = P_{N-1} - P_{N-1} C'_N C_N = P_{N-1} [I - C'_N C_N] . \qquad \text{(III-24)}$$