Two Dimensional Maximum Entropy Spectral Analysis

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We will restrict ourselves here to 2 dimensional spectra in k_x , k_y space. Higher dimensional spectra can be obtained using the same general procedures. We assume here that we have a 2 dimensional stationary process sampled at grid points given by $n\Delta x + m\Delta y$ where n and m are integers and Δx and Δy are the grid spacings in the x and y directions. The entropy of the spectrum $P(k_x, k_y)$ is given by

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \log [P(k_x, k_y)] \, dk_x \, dk_y \quad . \tag{1}$$

Our measurements or constraint equations are those specifying correlation values for particular 2-D lags.

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} P(k_x, k_y) \exp \left\{ 2\pi i \, n k_x \Delta x + 2\pi i \, m k_y \Delta y \right\} dk_x \, dk_y$$

$$= \phi_{nm} \tag{2}$$

By letting $z = \exp \left\{ 2\pi i \, k_x \Delta x \right\}$ and $w = \exp \left\{ 2\pi i \, k_y \Delta y \right\}$, we can rewrite (2) as

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} P(k_x, k_y) \, z^n \, w^m \, dk_x \, dk_y = \phi_{nm} \tag{3}$$

We now use Lagrange multipliers, λ_{nm} , to find the maximum entropy spectrum under the constraints (3).

$$\delta \iint [\log P(k_x, k_y) - \sum_{nm} \lambda_{nm} P(k_x, k_y) z^n w^m] \, dk_x \, dk_y$$

$$= \iint [P^{-1}(k_x, k_y) - \sum_{nm} \lambda_{nm} z^n w^m] \, \delta P(k_x, k_y) \, dk_x \, dk_y = 0 ,$$

or

$$P(k_x, k_y) = \frac{1}{\sum_{nm} \lambda_{nm} z^n w^m} . \tag{4}$$

Thus (4) is the functional form of the maximum entropy spectrum. The values of λ_{nm} are found by satisfying the constraint equations (3). Unfortunately, for the 2-D problem, a simple procedure of doing this is not known as it is for the single and multichannel cases. The following iterative technique is the best procedure known at present.

To illustrate the technique, we shall assume that our known correlation values are $\phi(0,0)$, $\phi(1,0)$, $\phi(1,1)$, $\phi(0,1)$ and $\phi(-1,1)$. Remembering that $\phi(n,m) = \phi(-n,-m)$, these values correspond to the following grid points in 2-D correlation space.

$$\begin{matrix} \phi(-1,1) & \phi(0,1) & \phi(1,1) \\ \phi(-1,0) & \phi(0,0) & \phi(1,0) \\ \phi(-1,-1) & \phi(0,-1) & \phi(1,-1) \end{matrix} .$$

For this special case, equation (4) can be written as

$$P(k_x, k_y) = [(\lambda_{-1,-1} w^{-1} + \lambda_{-1,0} + \lambda_{-1,1} w) z^{-1} + $$
$$+ (\lambda_{0,-1} w^{-1} + \lambda_{0,0} + \lambda_{0,1} w) + (\lambda_{1,-1} w^{-1} + \lambda_{1,0} + \lambda_{1,1} w) z]^{-1} . \tag{5}$$

We now recall that if we have an analytic function, $F(z)$, which we can expand on the unit circle as

$$F(z) = \sum_{n=-\infty}^{+\infty} a_n z^n , \tag{6}$$

then since $z = e^{i2\pi k_x \Delta x}$ and $dz = i2\pi \Delta x z \, dk_x$ or

$$dk_x = \frac{1}{i2\pi \Delta x} z^{-1} dz$$

we then have

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} F(z) z^{-s} dk_x = \frac{1}{2\pi i \Delta x} \oint F(z) z^{-s-1} dz =$$

$$\frac{1}{2\pi i \Delta x} \oint \sum_{n=-\infty}^{+\infty} a_n z^{n-s-1} dz = a_s / \Delta x .$$

This last equation arises because $\oint z^n dz = 0$ unless $n = -1$ in which case it is $2\pi i$. Our contour integral here is around the unit circle. The result of this paragraph is that

$$\Delta x \int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} F(z) z^{-n} dk_x = a_n \tag{7}$$

and that if we can expand $F(z)$ as (6) then we know the value of the integral (7) for all values of n.

Let us rewrite (5) as

$$P(k_x, k_y) = [A_1(w^{-1}) z^{-1} + A_0(w) + A_1(w) z]^{-1}, \tag{8}$$

where $A_1(w) = \lambda_{1,-1} w^{-1} + \lambda_{1,0} + \lambda_{1,1} w$ and

$$A_0(w) = \lambda_{0,-1} w^{-1} + \lambda_{0,0} + \lambda_{0,1} w .$$

Note that due to the symmetry of our correlation values, we will have $\lambda_{0\ -1} = \lambda_{0\ 1}$, $\lambda_{-1\ 1} = \lambda_{1\ -1}$, $\lambda_{-1\ 0} = \lambda_{1\ 0}$ and $\lambda_{-1\ -1} = \lambda_{1\ 1}$.
Now for a particular value of w, we can find the values of the integrals

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} P(k_x, k_y)\ z^n\ dk_x = S(k_y) \tag{9}$$

by expanding (8) in numerator form in z. We begin by factoring (8) into a minimum phase and a maximum phase filter in z, and then separate the expression into two parts.

$$P(k_x, k_y) = [(1 + Q(w^{-1}) z^{-1}) R^{-1}(w) (1 + Q(w) z]^{-1}$$

$$= \frac{R(w)}{2[1-Q(w^{-1})Q(w)]} \left\{ \frac{1 - Q(w^{-1})z^{-1}}{1 + Q(w^{-1})z^{-1}} + \frac{1 - Q(w)\ z}{1 + Q(w)\ z} \right\} . \tag{10}$$

Since $\dfrac{1 - Q(w)\ z}{1 + Q(w)\ z} = 1 - 2\ Q(w)\ z + 2\ Q^2(w)\ z^2 \ .....$

we see that for $n=0$,

$$S(k_y) = \frac{R(w)\ /\ \Delta x}{1 - Q(w^{-1})\ Q(w)} \quad ,$$

and for $n=-1$,

$$S(k_y) = \frac{-\ R(w)\ Q(w)\ /\ \Delta x}{1 - Q(w^{-1})\ Q(w)} \quad .$$

We now can numerically evaluate

$$\phi_{nm} = \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta x}} S(k_y)\ w^m\ d\,k_y$$

by the summation approximation of

$$\phi_{nm} \approx \frac{1}{2J\ \Delta y} \sum_{j=-J}^{J-1} S\left(\frac{j}{2J\Delta y} \right) \ \exp\left\{ \frac{\pi i\ m\ j}{J} \right\} \quad . \tag{11}$$

Thus, given values for λ_{nm} , we can numerically calculate values for ϕ_{nm} . Of course, what we want is to find the values for the λ_{nm} , given the $\phi(n,m)$. To do this, we used the solution procedure discussed in section, "The General Variational Approach."

From (3), plus the fact that $\phi_{nm} = \phi_{-n-m}$ we have

$$\frac{\partial\,\phi_{nm}}{\partial\,\lambda_{rs}} = \iint \frac{\partial\ P\ (\ k_x, k_y\ )}{\partial\ \lambda_{rs}}\ z^{-n}\ w^{-m}\ \ d\,k_x\ \ d\,k_y \quad .$$

Now using (4), we get

$$\frac{\partial \phi_{nm}}{\partial \lambda_{rs}} = - \iint \frac{z^{r-n} w^{s-m}}{\left[\sum_{uv} \lambda_{uv} z^{u} w^{v} \right]^{2}} \, d k_x \, d k_y \tag{12}$$

Letting

$$\psi_{r-n,s-m} = - \frac{\partial \phi_{nm}}{\partial \lambda_{rs}} , \tag{13}$$

we see that the ψ terms are autocorrelation values of $P^2(k_x, k_y)$. We can now write the changes in the ϕ_{nm} corresponding to the changes in the λ_{rs} as

$$\begin{Bmatrix} \Delta\phi_{-1-1} \\ \Delta\phi_{-1\,0} \\ \Delta\phi_{-1\,1} \\ \Delta\phi_{0-1} \\ \Delta\phi_{00} \\ \Delta\phi_{01} \\ \Delta\phi_{1-1} \\ \Delta\phi_{10} \\ \Delta\phi_{11} \end{Bmatrix} = - \begin{bmatrix} \psi_{00} & \psi_{01} & \psi_{02} & \psi_{10} & \psi_{11} & \psi_{12} & \psi_{20} & \psi_{21} & \psi_{22} \\ \psi_{10} & \psi_{00} & \psi_{01} & \psi_{1-1} & \psi_{10} & \psi_{11} & \psi_{2-1} & \psi_{20} & \psi_{21} \\ \psi_{20} & \psi_{01} & \psi_{00} & \psi_{1-2} & \psi_{1-1} & \psi_{10} & \psi_{2-2} & \psi_{2-1} & \psi_{20} \\ \psi_{-10} & \psi_{-11} & \psi_{-12} & \psi_{00} & \psi_{01} & \psi_{02} & \psi_{10} & \psi_{11} & \psi_{12} \\ \psi_{-1-1} & \psi_{-10} & \psi_{-11} & \psi_{10} & \psi_{00} & \psi_{01} & \psi_{1-1} & \psi_{10} & \psi_{11} \\ \psi_{-1-2} & \psi_{-1-1} & \psi_{-10} & \psi_{20} & \psi_{01} & \psi_{00} & \psi_{1-2} & \psi_{1-1} & \psi_{1-} \\ \psi_{-20} & \psi_{-21} & \psi_{-22} & \psi_{-10} & \psi_{-11} & \psi_{-12} & \psi_{00} & \psi_{01} & \psi_{02} \\ \psi_{-2-1} & \psi_{-20} & \psi_{-21} & \psi_{-1-1} & \psi_{-10} & \psi_{-11} & \psi_{10} & \psi_{00} & \psi_{01} \\ \psi_{-2-2} & \psi_{-2-1} & \psi_{-20} & \psi_{-1-2} & \psi_{-1-1} & \psi_{-10} & \psi_{20} & \psi_{01} & \psi_{00} \end{bmatrix} \begin{Bmatrix} \Delta\lambda_{-1-1} \\ \Delta\lambda_{-1\,0} \\ \Delta\lambda_{-1\,1} \\ \Delta\lambda_{0-1} \\ \Delta\lambda_{00} \\ \Delta\lambda_{01} \\ \lambda_{1-1} \\ \lambda_{1\,0} \\ \lambda_{1\,1} \end{Bmatrix} . \tag{14}$$

Equation (14) is in the convenient form of the multichannel matrix equation. In fact, (14) is even easier to solve than most multichannel problems since the 3 by 3 blocks are themselves symmetric about their minor diagonals. In this case, the forward and backward prediction error filters are the same under reversal in both time and space.

The iteration procedure is to start with a set of λ_{rs} which correspond to a positive spectrum but do not necessarily fit the constraint equations. The values for the ϕ'_{nm} corresponding to the λ_{rs} are then calculated by the procedure outlined in equations (5) through (11). The difference $\Delta\phi_{nm}$ between ϕ_{nm} and ϕ'_{nm} is then found and the changes in the λ_{rs} are then calculated by solving (14). The λ_{rs} are then corrected and the iteration continued until the overall error in satisfying the constraints reaches some specified minimum.

To calculate the autocorrelation values corresponding to the square of the spectrum, we can use (10) to get

$$P^2(k_x, k_y) = \frac{R^2(w)}{4\,[\,1-Q(w^{-1})\,Q(w)\,]^2}\left\{\left(\frac{1-Q(w^{-1})\,z^{-1}}{1+Q(w^{-1})\,z^{-1}}\right)^2 + 2\,\frac{1-Q(w^{-1})\,z^{-1}}{1+Q(w^{-1})\,z^{-1}}\cdot\frac{1-Q(w)\,z}{1+Q(w)\,z} + \left(\frac{1-Q(w)\,z}{1+Q(w)\,z}\right)^2\right\} .$$

The first and last terms of this expression are separated into minus and plus powers of z. The middle term is the autocorrelation of $P(k_x, k_y)$ convolved by $2\,(1-Q(w^{-1})z^{-1})(1-Q(w)z)$. Thus the

$S(k_y)$ can be obtained precisely from (10) in a few number of operations. Then equation (11) gives an approximation for λ_m .