An Example in Which Consistency is Equivalent to Positive Definiteness of the 2-D Correlation Matrix

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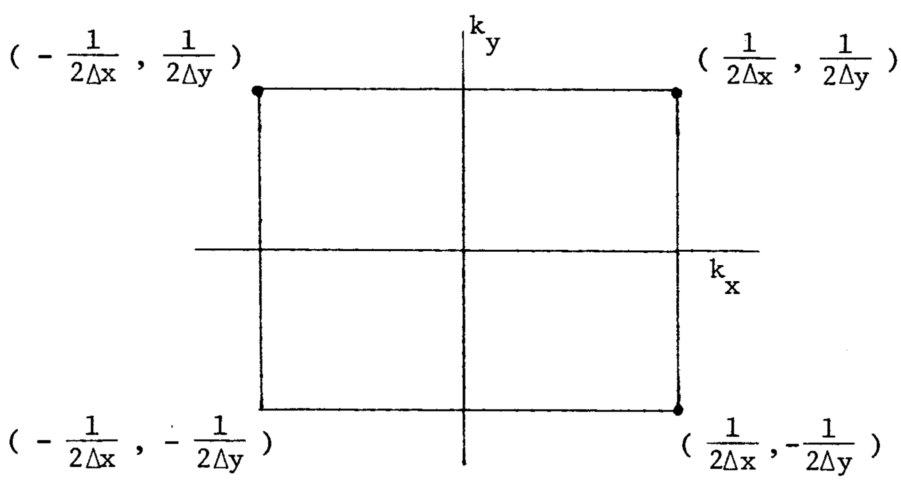
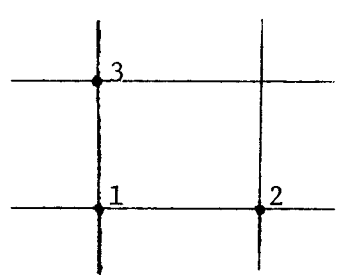
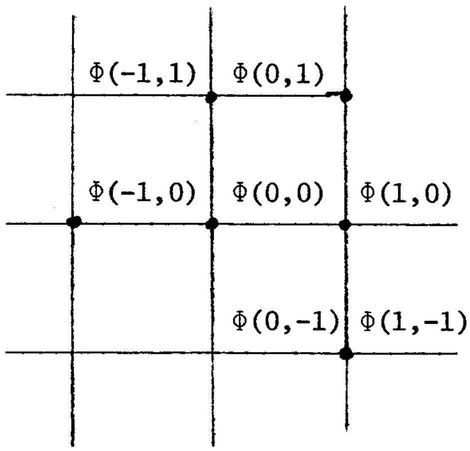
Suppose we have an infinite rectangular grid in the two dimensional x, y plane where the spacing between lines in the x direction is Δx and in the y direction is Δy. We assume that there is a two dimensional complex stationary random process, U_{st}, occurring in this plane and that it is sampled at the grid points. We further assume that the two dimensional spectra lies in the low frequency unit cell in k_x, k_y space. That is, the spectrum is zero outside of the rectangle given by $-\frac{1}{2\Delta x} \leqslant k_x \leqslant \frac{1}{2\Delta x}$ and $-\frac{1}{2\Delta y} \leqslant k_y \leqslant \frac{1}{2\Delta y}$, as shown below.



If we have a three point array on the x, y grid of the form



then if this array is passed through the stationary process, one could measure part of the 2-D auto correlation function by measuring the correlation between pairs of points in the three point array. We define the 2-D auto correlation function to be $\Phi(n,m)$ = Expected Value of $U_{s,t} \; U^*_{s+n\,,\,t+m}$. In correlation space, we can measure the following values of $(n\Delta x, m\Delta y)$.



If one were to consider the three point array as a pointer to a triplet of random variables from the stationary process, then the covariance matrix of this triplet would be

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} \Phi(0,0) & \Phi(1,0) & \Phi(0,1) \\ \Phi(-1,0) & \Phi(0,0) & \Phi(-1,1) \\ \Phi(0,-1) & \Phi(1,-1) & \Phi(0,0) \end{bmatrix} \end{array} = \text{Exp} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \begin{bmatrix} U_1^* & U_2^* & U_3^* \end{bmatrix}$$

It is known how to measure such a 3 by 3 covariance matrix from a finite set of triplets such that the matrix will be at least semi-positive definite and that the main diagonal terms are all equal, i.e., the estimated matrix will be of the correct form. The present question is whether or not semi-positive definiteness and being of the correct form are sufficient conditions for the covariance values to agree with some 2-D auto correlation function. That is, if the estimated matrix is

$$\begin{bmatrix} Q & A & C \\ A^* & Q & B \\ C^* & B^* & Q \end{bmatrix} \tag{1}$$

is there a 2-D correlation function that starts out with the values

$$\begin{matrix} B & C & \\ A^* & Q & A \\ & C^* & B^* \end{matrix}$$

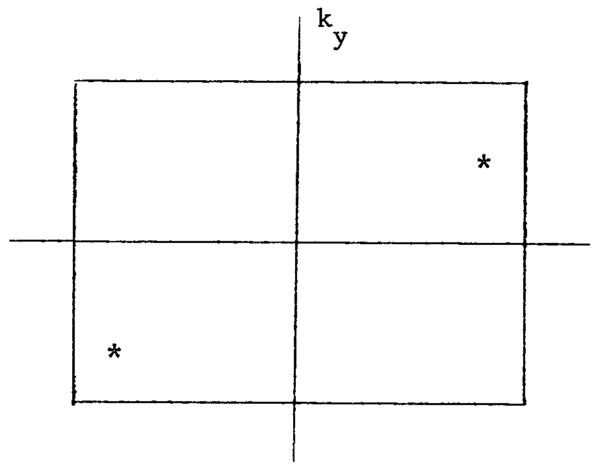
For this particular geometry the answer is yes. We will prove this by constructing a spectrum with these auto correlation values. In particular, the spectrum will be made up of white noise plus two delta functions in the k_x , k_y unit cell.

Since (1) is at least semi-positive definite, the eigenvalues are non-negative. If we substract the smallest eigenvalue from the main diagonal, we will then have a singular matrix. Without loss of generality, we will assume that after this is done, our matrix has ones along the main diagonal, i.e.,

$$\begin{bmatrix} 1 & A & C \\ A^* & 1 & B \\ C^* & B^* & 1 \end{bmatrix} . \tag{2}$$

We have in effect removed white noise of power $Q - 1$ from the matrix. We will now show that the rest of the correlation function is in agreement with a spectrum consisting of two delta functions in the k_x , k_y unit cell.

Because it is much simpler, we will only give the solution for a real 2-D stationary process. In this case, A, B and C are real and the two delta functions must be equal in power and symmetrically placed with respect to the origin in k_x , k_y space. Actually, the solution is that one half of the power is located at the point $k_x = h_x$, $k_y = h_y$ such that $\cos 2\pi\, h_x\, \Delta x = A$ and $\cos 2\pi\, h_y\, \Delta y = C$, with the other half of the power at the symmetrically placed point.



The fourier transform relations between a spectrum and its correlation function are

$$P(k_x,k_y) = \frac{1}{\Delta x \Delta y} \sum_{n,m} \Phi(n,m)^{\exp[-\, i2\pi(\, k_x\, n\Delta x + k_y\, m\Delta y\,)]}$$

and

$$\Phi(n,m) = \int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} P(k_x,k_y)^{\exp[\, i2\pi(k_x n\Delta x + k_y m\Delta y)]}\, dk_x\, dk_y \, .$$

For our stated solution, the spectrum is

$$P(k_x, k_y) = \frac{1}{2}\delta(k_x - h_x, k_y - h_y) + \frac{1}{2}\delta(k_x + h_x, k_y + h_y)$$

and two of the corresponding correlation function values are

$$A = \Phi(1,0) = \iint [\tfrac{1}{2}\delta(k_x - h_x, k_y - h_y) + \tfrac{1}{2}\delta(k_x + h_x, k_y + h_y)]$$

$$\cdot \exp[i2\pi k_x \Delta x]\, d k_x\, d k_y = \frac{1}{2}\exp[i2\pi k_x \Delta x] + \frac{1}{2}\exp[-i2\pi h_x \Delta x] =$$

$$= \cos 2\pi h_x \Delta x = A$$

and

$$C = \Phi(0,1) = \iint [\tfrac{1}{2}\delta(k_x - h_x, k_y - h_y) + \tfrac{1}{2}\delta(k_x + h_x, k_y + h_y)]$$

$$\cdot \exp[i2\pi k_y \Delta y]\, d k_x\, d k_y = \frac{1}{2}\exp[i2\pi h_y \Delta y] + \frac{1}{2}\exp[-i2\pi h_y \Delta y]$$

$$= \cos 2\pi h_y \Delta y = C .$$

Thus, this spectrum agrees with $\Phi(1,0)$ and $\Phi(0,1)$. To see about $\Phi(-1,1)$, we note that

$$B = \Phi(-1,1) = \iint [\tfrac{1}{2}\delta(k_x - h_x, k_y - h_y) + \tfrac{1}{2}\delta(k_x + h_x, k_y + h_y)]$$

$$\cdot \exp[-i2\pi k_x \Delta x + i2\pi k_y \Delta_y]\, d k_x\, d k_y$$

$$= \frac{1}{2}\exp[-i2\pi h_x \Delta x + i2\pi h_y \Delta y] + \frac{1}{2}\exp[i2\pi h_x \Delta x - i2\pi h_y \Delta y]$$

$$= \cos[2\pi h_x \Delta x - 2\pi h_y \Delta y] = \cos(2\pi h_x \Delta x)\cos(2\pi h_y \Delta y)$$

$$+ \sin(2\pi h_x \Delta x)\sin(2\pi h_y \Delta y) = AC \pm [(1-A^2)(1-C^2)]^{\frac{1}{2}} . \qquad (3)$$

Thus we need

$$(B-AC)^2 = (1-A^2)(1-C^2) ,$$

or for $1 - A^2 - B^2 - C^2 + 2\,ABC = 0$.

However, this is simply the determinant of our singular matrix and thus is a valid relation.

In equation (3), the $\pm$ indicates that our specification of the location of the delta functions was incomplete since we did not determine the signs of h_x and h_y . Their unambiguous locations are pinned down by choosing the correct sign for the equation (3).

The conclusion is that if we have an array of three non-colinear points, then the corresponding correlation measurements will be consistent if and only if the 3 by 3 covariance matrix is non-negative definite. This is not necessarily true for three colinear array points or for two dimensional arrays of four or more points.