Different Approximations to Vertical Derivative Operators in Potential Problems

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With the study of magnetic anomaly profiles in mind, Le Mouël et al (1974) developed a simple analytic operator that allows one to compute the continuation of a potential field recorded on a contour of any shape. We shall only deal here with the two dimensional problem and limit ourselves to a potential field recorded on a horizontal line, taken to be parallel to axis Ox. The vertical axis is positive upwards and called Oy (and not Oz, since we will need z for other purposes). We will take the data sampling interval as unity (Δx = constant = 1). Let $u_k = u(k, 0)$ be the set of data at integer values of x, for $y = 0$. The continuation formula of Le Mouël et al (1974) is:

$$u(P) = \sum_{k=-\infty}^{+\infty} u_k \cdot \operatorname{Im} \left\{ \frac{e^{i\pi(z-k)} - 1}{\pi(z-k)} \right\} \qquad (1)$$

where $P(x,y)$ - with $z = x + iy$ - is the point where continuation is desired; this formula can be used for upward or downward continuation, but will only be meaningful as long as one does not go below the sources of the potential field u. (1) is essentially a convolution equation where the infinite series $\{u_k\}$ is convolved by:

$$a(z) = \operatorname{Im} \left\{ \frac{e^{i\pi z} - 1}{\pi z} \right\} \qquad (2)$$

This function is analytic, reduces to the well-known sinc x function on the Ox axis and goes to 0 as y goes to $+\infty$. Thus $u(P)$ is an analytic function of z and can be differenciated with respect to either x or y . In particular, vertical derivatives up to any order can be obtained; in the case of the first vertical derivative, it is easy to show that:

$$\partial_y u = \sum_{k=-\infty}^{+\infty} u_k \cdot a'_k \quad (\text{ at } x = y = 0) \tag{3}$$

with:

$$a'_0 = -\pi/2 \ , \ a'_k = [\ 1 - (-1)^k\ ] \ / \ \pi k^2 \quad \text{for } k \neq 0 \tag{4}$$

With the following definition for the Fourier transform:

$$\begin{cases} G(f) = \sum_k a_k\, e^{-2i\pi fk} \ , & |f| < 1/2 \ ; \ G(f) = 0 \ , \ |f| > 1/2 \\ a_k = \int_{-1/2}^{1/2} G(f)\, e^{2i\pi fk}\, df \end{cases} \tag{5}$$

it is easy to show that the Fourier series for the a'_k is $2\pi|f|$. If we substitute the wave number k_x for $2\pi f$ and introduce the Z variable e^{-ik_x} (this Z should not be mistaken for the $Z = x + iy$ corresponding to point P), we get the following result:

$$\sum_k a'_k\, Z^k = |k_x| \tag{6}$$

When evaluating $|k_x|$ for computational purposes, the Z-transform in (6) will have to be truncated. In order to have an idea of the relative error due to this truncation, one need only notice that a'_k

converges towards zero as k^{-2} and that every other term is zero.

We can next obtain an expression similar to (3,4) for the second vertical derivative:

$$\partial_{yy} u = \sum_k u_k \cdot a''_k \quad (\text{at} \quad x = y = 0) \tag{7}$$

with:

$$a''_0 = \pi^2/3 \ , \ a''_k = 2 \ (-1)^k / k^2 \quad \text{for} \quad k \neq 0 \tag{8}$$

a''_k converges as $(-1)^k / k^2$, which means that with 10 terms in (7), one will obtain an estimate of $k_x^{\ 2}$ with a relative error on the order of 1%.

This can also be obtained by convolving the first vertical derivative operator by itself, thus getting the Z-transform expression for $k_x^{\ 2} = |2\pi f|^2$:

$$\sum_k a''_k \cdot Z^k = k_x^{\ 2} = \partial_{yy} \tag{9}$$

If we remember that we are dealing with potential functions ($\partial_{xx} + \partial_{yy} = 0$) we find, as could have been expected:

$$k_x^{\ 2} = - \partial_{xx} \tag{10}$$

All operators considered here are <u>exact</u>, when the infinite series $\{u_k\}$ is used and when the potential function u is band limited in the (- 1/2 , 1/2) range. If one of these assumptions fails, one only has an approximate expression for the Z-transform of the operator. The quality of this approximation can be judged from examples in Le Mouël et al (1974).

Now, we can rewrite (10) as:

$$\partial_{xx} = -\frac{\pi^2}{3} + \sum_{k\neq 0} \frac{2(-1)^{k+1}}{k^2} \cdot Z^k \qquad (11)$$

This expansion can be compared with one obtained from the following relationship between ∂_x , the differenciation operator, and δ_x , the corresponding finite difference operator (Mitchell, p. 18)

$$\partial_x = 2 \text{ arg sinh } \frac{\delta_x}{2} = \delta_x - \frac{1^2}{2^2\, 3!} \delta_x^{\ 3} + \frac{1^2\, 3^2}{2^4\, 5!} \delta_x^{\ 5} + \cdots \qquad (12)$$

By squaring this expression, one gets:

$$\partial_{xx} = \delta_{xx} - \frac{1}{12} \delta_x^{\ 4} + \frac{1}{90} \delta_x^{\ 6} + \cdots \qquad (13)$$

By replacing $\delta_x^{\ 2}$ by $(Z^{-1} - 2 + Z)$ in (13), one hopefully finds the coefficients of powers of Z in (13) to be identical to those in (11). The usual procedure, when solving PDE such as the wave equation, is to approximate ∂_{xx} with a truncation of (13) rather than (11); the one-term and two-term truncations of (13) are the most common ones. It may be of some interest to notice that when truncating (13) and expressing it as a Z-transform $\sum_{k=-N}^{N} \alpha''_k \cdot Z^k$, the α''_k are different from the a''_k (of course they converge towards the a''_k when N goes to infinity). The difference between (11) and (13) is somewhat similar to the difference that exists between the expansion of a function on an orthogonal set and on a non-orthogonal one; in the latter case, the coefficients in a truncated form of the expansion vary with the order N of the truncation. It is not obvious to me, right now, which truncation is to be preferred: the preceding remark would tend to favor (11) where the a''_k are exact, but use of (13) in previous SERP reports and in Claerbout's papers obviously gave very satisfactory results.

Let us now return to the first vertical derivative operator; equation (3) is one way of computing it but other ways can be found: one may, for example, look for the roots of the Laplace equation $\partial_{xx} + \partial_{yy} = 0$, which can be formally written in the following way:

$$\partial_y = \pm i \partial_x \tag{14}$$

When applying these two possible operators to a complex potential function $\psi = u + iv$, and equating the real parts, one gets:

$$\partial_y u = \mp \partial_x v \tag{15}$$

Thus, in order to compute $\partial_y u$ from the knowledge of the (u_k) , one can first compute v from u through a Hilbert transform operation (this is simply a convolution with $-1/\pi x$, which can be computed with the FFT algorithm), and then compute $\partial_y u$ from (15), using any desirable approximation of ∂_x (see (12)).

Of course, v can also be directly obtained from (1) through:

$$v(P) = \sum_{k=-\infty}^{+\infty} -u_k \cdot \operatorname{Re}\left\{\frac{e^{i\pi(z-k)} - 1}{\pi(z-k)}\right\} \tag{16}$$

It is easy to see that the use of (16) and of the exact ∂_x in (15) lead exactly to the single equation (3). In (3) the coupling between u and v is not apparent, while it is apparent in (15).

<u>Remark</u>: the two possible solutions in (15) correspond to potential fields going to zero either as $y \to +\infty$ (sources in the lower half-space, which has been assumed in order to be able to write continuation

equation (1)) or as $y \to -\infty$ (sources in the upper half-space); this is somewhat reminiscent of the separation between up and downgoing waves in the case of the wave equation. One might hope to be able to use the various approaches described in this note in order to obtain different expansions and approximations of ∂_y in the case of the wave equation. Unfortunately, it does not seem possible to find an equivalent of (3) for that case. The wave equation for a monochromatic wave is:

$$\partial_{xx} + \partial_{yy} = -\omega^2 c^{-2} \tag{17}$$

which leads to the following solutions for ∂_y :

$$\begin{aligned} \partial_y &= \pm\, i \frac{\omega}{c} \left(1 + \frac{c^2}{\omega^2} \partial_{xx} \right)^{1/2} \\ &\approx \pm\, i \frac{\omega}{c} \left(1 + \frac{c^2}{2\omega^2} \partial_{xx} + \cdots \right) \end{aligned} \tag{18}$$

which, when applied to $\psi = u + iv$, leads to the following equivalent of equation (15):

$$\partial_y u = \mp \frac{\omega}{c} \left(1 + \frac{c^2}{2\omega^2} \partial_{xx} + \cdots \right) v \tag{19}$$

which can be approximated by:

$$\partial_y u \simeq \mp \frac{\omega}{c} v \mp \frac{c}{2\omega} \delta_{xx} v \tag{20}$$

Thus only a "coupled" expansion analogous to (15) can be found, which has already been widely used in the SERP report and in Claerbout's papers.

Reference

LeMouel, J. L., Courtillot, V. E., and Galdeano, A., 1974, A simple formalism for the study of transformed aeromagnetic profiles and source location problems, J. Geophys. Res., 79, 324-331.