# An Explicit Scheme for $P_{zt} = \frac{v}{2} P_{xx}$

by Jon F. Claerbout

We consider the differential equation

$$P_{zt} = \frac{v}{2} P_{xx} \tag{1}$$

We have the difference representation

$$\delta_z \; \delta_t \; P = \frac{v \; \Delta t \; \Delta z}{2 \; \Delta x^2} \; \delta_{xx} \; P \tag{2}$$

Following Riley (SEP March '74, p. 61) we define

$$a = \frac{v \; \Delta z \; \Delta t}{8 \; \Delta x^2} \tag{3}$$

Using (3) and Mitchell's higher order scheme (Computational Methods in Partial Differential Equations, A. R. Mitchell, page 26) equation (2) is replaced by

$$\delta_z \; \delta_t \; P = 4 \; a \; \frac{\delta_{xx}}{1 + b \; \delta_{xx}} \; P \tag{4}$$

Mitchell shows that $b = 1/12$ gives 4th order accuracy. Ever since I discovered this trick I used $1/12$ because seismic data is usually undersampled in space. J. W. C. Sherwood mentioned at our April 1974 meeting that $b = 1/6$ is only 2nd order accurate but fits over a wider range than $b = 1/12$ .

Letting $-\delta_{xx}$ be denoted by a tri-diagonal matrix $T$ with ( -1, 2, -1 ) on the main diagonal (4) becomes

$$( 1 - b T )( P_{k+1}^{n+1} + P_k^n - P_k^{n+1} - P_{k+1}^n ) = - a T ( P_{k+1}^{n+1} + P_k^n + P_k^{n+1} + P_{k+1}^n ) \quad (5)$$

bringing the unknown to the left

$$( 1 + (a-b) T ) P_{k+1}^{n+1} = P_k^{n+1} + P_{k+1}^n - P_k^n - (a-b) T - (a+b) T ( P_k^{n+1} + P_{k+1}^n ) \quad (6)$$

If it should happen that $a = b$ then (6) reduces to

$$P_{k+1}^{n+1} = ( P_k^{n+1} + P_{k+1}^n ) - P_k^n - 2 a T ( P_k^{n+1} + P_{k+1}^n ) \quad (7)$$

From a computational point of view (7) is a drastic simplification over (6) because it is explicit. A test revealed that in Riley's benchmark program (SEP, March 74, page 64) it was stable and ran 2.5 times faster for the same number of steps. However, Riley ran with an $a = 1$ whereas to be consistent with Mitchell or Sherwood it was necessary to run with $a = 1/12$ or $a = 1/6$. This amounts to a considerable reduction of $\Delta z$ and raises the question of how we should choose $\Delta z$ given that $\Delta x$ and $\Delta t$ have already been fixed. Instead of answering the question of how $\Delta z$ should be chosen, I looked at a number of examples done by myself and others and found that numerical values for $a$ were indeed commonly in the range from 1/6 to 1/12. A replacement for Riley's FAST15 subroutine is attached.

Stability limits on $a$ and $b$ remain to be determined. It is clear that (6) will get in trouble when zero eigenvalues of $( I + (a-b) T )$ become possible. The extreme eigenvalues of $T$ are achieved with the eigenvectors ( 1, 1, 1, 1, ... 1 ) and ( 1, -1, 1, -1, 1, ... ). Since the eigenvalues of $T$ range from

0 to 4 we see that (6) requires $(a-b) > -1/4$ . This means that if $b$ is chosen greater than $1/4$ there will exist a small $\Delta z$ for which (6) is unstable. There seems to be no problem because no one has yet advocated that big a choice of $b$ .

To study the stability of (7) we use z-transform theory. If $P^{n+1}$ and $P^n$ are regarded as z-transforms then the following equation reduces to (7) on identification of the coefficient of $z^k$

$$z^{-1} P^{n+1} = ( P^{n+1} + z^{-1} P^n ) - P^n - 2aT ( P^{n+1} + z^{-1} P^n ) \tag{8}$$

or

$$( 1/z - 1 + 2aT ) P^{n+1} = ( 1/z - 1 - 2aT / z ) P^n$$

which gives us the transfer function

$$P^{n+1} / P^n = \frac{( 1 - 2aT ) - z}{1 - z ( 1 - 2aT )} \tag{9}$$

The stability for the depth recursion is ascertained by checking to see whether the modulus of (9) on the unit circle is unity or less. Letting $1 - 2aT = \rho$ the modulus of (9) is written

$$\frac{( \rho - 1/z )( \rho - z )}{( 1 - \rho/z )( 1 - \rho z )} = 1$$

Thus, the depth recursion is unitary, an ideal state of affairs.

Finally we must check into the time recursion. For stability in the forward (backward) direction in time it is essential that the pole of (9) be outside (inside) the unit circle in the z-plane for all of the