Migration Equation Coefficients for An Emergent Angle Frame in Stratified Media

by Raul Estevez

In his paper of July 3, Claerbout described two different frames based on a layered model and pointed out the necessity of computing numerically the coefficients of the corresponding migration equations. For a given velocity model $v(z)$, this would imply the computation of several tables connecting the travel time τ , the horizontal displacement u , the depth w and the ray parameter $p = \sin\theta/v$. Specifically, we would have to compute $u(p,w)$, $\tau(p,w)$ and $p(u,w)$.

In the present paper we intend to accomplish this task for the first frame (p - frame). The remaining frame (h - frame) will be considered in a later paper.

We start from a coordinate system, with s , z , g and t as independent variables (Fig. 1).

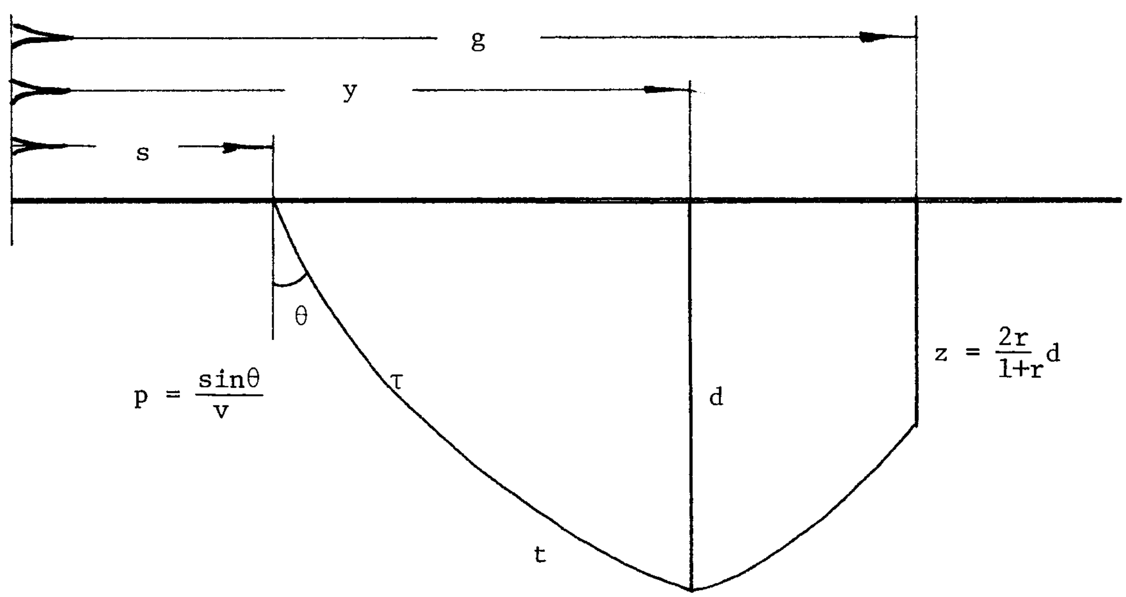

FIG. 1

The transformed variables p, y, d, r are then defined by the inverse of the following transformation:

$$s(p, y, d, r) = y - u(p,d) \quad (1\text{-}1a)$$

$$z(p, y, d, r) = 2rd/(1+r) \quad (1\text{-}1b)$$

$$g(p, y, d, r) = y + u(p,d) - u(p, 2rd/(1+r)) \quad (1\text{-}1c)$$

$$t(p, y, d, r) = 2\tau(p,d) - \tau(p, 2rd/(1+r) \quad (1\text{-}1d)$$

Now let's try to compute $u(p,w)$ and $\tau(p,w)$ for a layered media.

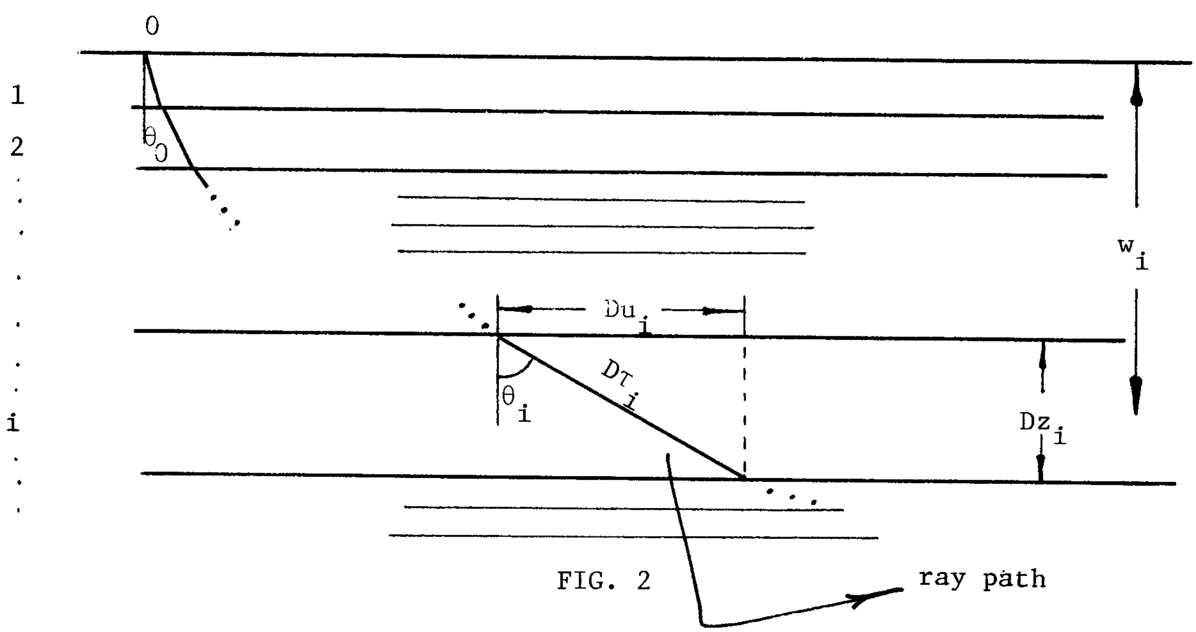

FIG. 2

According to Fig. 2, we have the following relations for the i^{th} layer:

$$Du_i = Dz_i \tan\theta_i \quad (1\text{-}2)$$

$$Dz_i = v_i D\tau_i \cos\theta_i \quad (1\text{-}3)$$

$$\sin\theta_i = pv_i \quad (1\text{-}4)$$

Therefore, from (1-2) and (1-4), we find:

$$D\, u_i = D\, z_i \sin\theta_i \,/\, (1-\sin^2\theta_i)^{1/2} = p\, v_i [\, 1-(pv_i)^2]^{-1/2}\, D\, z_i$$

Whence we obtain for the case of a continuous variation of v with depth:

$$u(p,w) = p\int_0^w v(z)\; [\; 1 - (pv(z))^2\,]^{-1/2}\; dz \qquad (1\text{-}5)$$

In the other hand, from (1-3) and (1-4) we have

$$D\, \tau_i = \frac{D\, z_i}{v_i \cos\theta_i} = \frac{D\, z_i}{v_i [1-(pv_i)^2]^{1/2}} \quad ,$$

whence

$$\tau(p,w) = \int_0^w \frac{1}{v(z)} [\; 1 - (pv(z))^2]^{-1/2}\; dz \qquad (1\text{-}6)$$

Substituting (1-5) and (1-6) into (1-1), we get for the direct transformation:

$$s = y - \int_0^d pv(z) [\; 1 - (pv)^2\;]^{-1/2}\; dz \qquad (1\text{-}7a)$$

$$z = 2rd \,/\, (1+r) \qquad (1\text{-}7b)$$

$$g = y + \int_0^d pv(z) [1-(pv)^2]^{-1/2} dz - \int_0^{2rd/(1+r)} pv[1-(pv)^2]^{-1/2}\; dz =$$

$$= y + \int_{2rd/(1+r)}^{d} pv[1 - (pv)^2]^{-1/2}\; dz \qquad (1\text{-}7c)$$

$$t = 2\int_0^d \frac{1}{v(z)} [1-(pv)^2]^{-1/2}\; dz - \int_0^{2rd/(1+r)} \frac{1}{v(z)} [1-(pv)^2]^{-1/2}\; dz =$$

$$= \int_0^{2rd/(1+r)} \frac{1}{v}[1-(pv)^2]^{-1/2} + 2\int_{2rd/(1+r)}^{d} \frac{1}{v}[1-(pv)^2]^{-1/2}\; dz \qquad (1\text{-}7d)$$

Now, since we are interested in the inverse of this transformation, what we really want is the Jacobian:

$$\begin{bmatrix} s_p & s_y & s_d & s_r \\ z_p & z_y & z_d & z_r \\ g_p & g_y & g_d & g_r \\ t_p & t_y & t_d & t_r \end{bmatrix} = \begin{bmatrix} s_p & 1 & s_d & 0 \\ 0 & 0 & z_d & z_r \\ g_p & 1 & g_d & g_r \\ t_p & 0 & t_d & t_r \end{bmatrix} \tag{1-8}$$

Calling $d' = 2rd/(1+r)$, the remaining derivatives in this Jacobian are:

$$s_p = -\int_0^d v[\ 1 - (pv)^2\ ]^{-1/2}\ dz - p^2\int_0^d v^3[\ 1 - (pv)^2\ ]^{-3/2}\ dz \tag{1-9a}$$

$$s_d = -\ pv(d)[\ 1 - (pv(d))^2\ ]^{-1/2} \tag{1-9b}$$

$$z_d = 2r/(1+r) \tag{1-10a}$$

$$z_r - 2d/(1+r)^2 \tag{1-10b}$$

$$g_p = \int_{d'}^d v[\ 1 - (pv)^2]^{-1/2} dz + p^2\int_{d'}^d v^3[\ 1 - (pv)^2]^{-3/2}\ dz \tag{1-11a}$$

$$g_d = pv(d)[1-(pv(d))^2]^{-1/2} - \frac{2r}{1+r}\ pv(d')\ [1-(pv(d'))^2]^{-1/2} \tag{1-11b}$$

$$g_r = -\frac{2d}{(1+r)^2}\ pv(d')\ [\ 1 - (pv(d'))^2\ ]^{-1/2} \tag{1-11c}$$

$$t_p = \int_0^{d'} pv[1-(pv)^2]^{-3/2}\ dz + 2\int_{d'}^d pv[1-(pv)^2]^{-3/2}\ dz \tag{1-12a}$$

$$t_d = \frac{2}{v(d)}[1-(pv(d))^2]^{-1/2} - \frac{2r}{1+r}\ \frac{1}{v(d')}[1-(pv(d'))^2\ ]^{-1/2} \tag{1-12b}$$

$$t_r = -\frac{2d}{(1+r)^2}\ \frac{1}{v(d')}[\ 1 - (pv(d'))^2\ ]^{-1/2} \tag{1-12c}$$

If we now introduce the following notations:

$$I_1(p,d) = \int_0^d v[\ 1 - (pv)^2\ ]^{-1/2}\ dz \qquad (1\text{-}13a)$$

$$I_2(p,d) = \int_{d'}^d v[\ 1 - (pv)^2\ ]^{-1/2}\ dz \qquad (1\text{-}13b)$$

$$I_3(p,d) = \int_0^d v^3[\ 1 - (pv)^2\ ]^{-3/2}\ dz \qquad (1\text{-}13c)$$

$$I_4(p,d) = \int_{d'}^d v^3[1 - (pv)^2\ ]^{-3/2}\ dz \qquad (1\text{-}13d)$$

$$I_5(p,d) = \int_0^{d'} v[\ 1 - (pv)^2\ ]^{-3/2}\ dz \qquad (1\text{-}13e)$$

$$I_6(p,d) = \int_{d'}^d v[\ 1 - (pv)^2\ ]^{-3/2}\ dz \qquad (1\text{-}13f)$$

$$SQ1(p,d) = [\ 1 - (pv(d))^2\ ]^{-1/2} \qquad (1\text{-}13g)$$

$$SQ2(p,d,r) = [\ 1 -(pv(d'))^2\ ]^{-1/2}\ , \qquad (1\text{-}13h)$$

the elements of the Jacobian of the direct transformation can be written as:

$$s_p = -I_1 - p^2\ I_3 \qquad (1\text{-}14a)$$

$$s_y = 1 \qquad (1\text{-}14b)$$

$$s_d = -\ pv(d)\ SQ1 \qquad (1\text{-}14c)$$

$$s_r = 0\ . \qquad (1\text{-}14d)$$

$$z_p = 0 \tag{1-15a}$$

$$z_y = 0 \tag{1-15b}$$

$$z_d = 2r/(1+r) \tag{1-15c}$$

$$z_r = 2d/(1+r)^2 \tag{1-15d}$$

$$g_p = I_2 + p^2 I_4 \tag{1-16a}$$

$$g_y = 1 \tag{1-16b}$$

$$g_d = s_d - z_d \, pv(d') \, SQ2 \tag{1-16c}$$

$$g_r = - z_r \, pv(d') \, SQ2 \tag{1-16d}$$

$$t_p = p(I_5 + 2 I_6) \tag{1-17a}$$

$$t_y = 0$$

$$t_d = \frac{2}{v(d)} SQ1 - z_d \frac{1}{v(d')} SQ2 \tag{1-17c}$$

$$t_r = - z_r \frac{1}{v(d')} SQ2 \tag{1-17d}$$

The computation of this Jacobian and its inverse as functions of p, d, r and for a given velocity model $v(z)$, shouldn't be a difficult or expensive task. Once we have computed the Jacobian of the inverse transformation, the coefficients of the transformed wave equation can easily be found. Apparently we have to compute 6 different integrals, (I_1 to I_6), but in practice we only have 3 integrals, which can be computed together with the functions SQ1 and SQ2 quite cheaply. This will be shown later.

The considered frame differs from the one studied by Steve Doherty in his paper of June 17, in that $z(p,y,d,r)$ now equals $2rd/(1+r)$. As a basis for further reference and checking, we solve the above problem for two different velocity-models: constant velocity and velocity linearly varying with depth.

Constant velocity model:

If the velocity is constant throughout the medium, the expressions for $u(p,d)$ and $\tau(p,d)$ ((1-5) and (1-6)), simply become

$$u(p,d) = pvd[\, 1 - (pv)^2 \,]^{-1/2} \qquad (2\text{-}1)$$

and

$$\tau(p,d) = (d/v)\,[\, 1 - (pv)^2 \,]^{-1/2} \quad , \qquad (2\text{-}2)$$

therefore, the transformation (1-1) can be rewritten as:

$$s = y - pvd[\, 1 - (pv)^2 \,]^{-1/2} \qquad (2\text{-}3a)$$

$$z = 2\, rd/(1+r) \qquad (2\text{-}3b)$$

$$g = y + \frac{1-r}{1+r}\, pv\, d[\, 1 - (pv)^2 \,]^{-1/2} \qquad (2\text{-}3c)$$

$$t = \frac{2}{1+r}\, \frac{d}{v}\, [\, 1 - (pv)^2 \,]^{-1/2} \qquad (2\text{-}3d)$$

The Jacobian corresponding to this transformation, equivalent to (1-8), can easily be computed:

$$s_{p;y;d;r} = -\, vd[1-(pv)^2]^{-3/2} \; ; \; 1 \; ; \; -\, pv[1-(pv)^2]^{-1/2} \; ; \; 0 \qquad (2\text{-}4a)$$

$$z_{p;y;d;r} = 0 \; ; \; 0 \; ; \; 2r/(1+r) \; ; \; 2d/(1+r)^2 \qquad (2\text{-}4b)$$

$$g_{p;y;d;r} = \frac{1-r}{1+r}\, vd[1-(pv)^2]^{-3/2} \; ; \; 1 \; ; \; \frac{1-r}{1+r}\, pv[1-(pv)^2]^{-1/2} \; ; \; -\frac{2}{(1+r)^2}\, pvd[1-(pv)^2]^{-1/2} \qquad (2\text{-}4c)$$

$$t_{p;y;d;r} = \frac{2}{1+r}\, pdv[1-(pv)^2]^{-3/2} \; ; \; 0 \; ; \; \frac{2}{1+r}\,\frac{1}{v}[1-(pv)^2]^{-1/2} \; ; \; -\frac{2}{(1+r)^2}\,\frac{d}{v}\,[1-(pv)^2]^{-1/2} \qquad (2\text{-}4d)$$

The transformation, inverse to (2-3), can also be computed through lengthy but not difficult algebra. If, following Steve, we write

$$\beta(s,g,t) = [\,(vt)^2 - (g-s)^2\,]^{1/2} \;, \qquad (2\text{-}5)$$

the inverse transformation is found to be:

$$p(s,z,g,t) = (g-s)\,/\,v^2t \qquad (2\text{-}6a)$$

$$y(s,z,g,t) = (g+s)/2 + z(g-s)/2\,\beta \qquad (2\text{-}6b)$$

$$d(s,z,g,t) = (\beta+z)/d \qquad (2\text{-}6c)$$

$$r(s,z,g,t) = z/\beta \qquad (2\text{-}6d)$$

Its Jacobian is then:

$$p_{s;z;g;t} = -1/v^2t \ ; \ 0 \ ; \ 1/v^2t \ ; \ -(g-s)/v^2t^2 \qquad (2\text{-}7a)$$

$$y_{s;z;g;t} = \frac{1}{2}(1 - \frac{z(vt)^2}{\beta^3}); \ \frac{g-s}{2\ \beta} \ ; \ \frac{1}{2}(1 + \frac{z(vt)^2}{\beta^3}); \ -\frac{1}{2\ \beta^3} z\ v^2\ t(g-s) \qquad (2\text{-}7b)$$

$$d_{s;z;g;t} = (g-s)/2\beta \ ; \ 1/2 \ ; \ -(g-s)/2\beta \ ; \ v^2t/2\beta \qquad (2\text{-}7c)$$

$$r_{s;z;g;t} = -z(g-s)/\beta^3 \ ; \ 1/\beta \ ; \ z(g-s)/\beta^3 \ ; \ -z\ v^2t/\beta^3 \qquad (2\text{-}7d)$$

If we now insert these elements into the transformed wave equation:

$$[(p_g\partial_p + y_g\partial_y + d_g\partial_d + r_g\partial_r)^2 + (p_z\partial_p + y_z\partial_y + d_z\partial_d + r_z\partial_r)^2 - $$

$$-\frac{1}{v^2}(p_t\partial_t + y_t\partial_t + d_t\partial_t + r_t\partial_t)^2\ ]\ Q = 0\ , \qquad (2\text{-}8)$$

we get as expected, zero coefficients for Q_{dd} , Q_{rp} , Q_{dp} and Q_{yd} . The remaining coefficients of interest are:

Coefficient of Q_{pp} :

$$p_g^2 + p_z^2 - \frac{1}{v^2}p_t^2 = \frac{1}{v^4t^2}[1 - \frac{(g-s)^2}{(vt)^2}] = \frac{\beta^2}{v^6t^4} = \frac{(1+r)^2}{4v^2d^2}[1 - (pv)^2]^2 \qquad (2\text{-}9a)$$

Coefficient of Q_{rr} :

$$r_g^2 + r_z^2 - \frac{1}{v^2}\ r_t^2 = \frac{1}{\beta^2} - \frac{z^2}{\beta^6}[(g-s)^2 - (vt)^2] = \frac{1}{\beta^2}(1-r^2) = \frac{(1-r)(1+r)^3}{4\ d^2} \qquad (2\text{-}9b)$$

Coefficient of Q_{rd} :

$$2(r_gd_g + r_zd_z - \frac{1}{v^2}\ r_td_t) = 2(-\frac{1}{2}\ \frac{z(g-s)^2}{\beta^4} + \frac{1}{2\beta} + \frac{z(vt)^2}{2\ \beta^4}) = \frac{2d}{\beta^2} = \frac{(1+r)^2}{2d} \qquad (2\text{-}9c)$$

Coefficient of Q_{yy} :

$$y_g^2 + y_z^2 - \frac{1}{v^2} y_t^2 = \frac{1}{4} \frac{(1+r)^2}{1-(pv)^2} \qquad (2\text{-}9d)$$

Coefficient of Q_{yp} :

$$2(p_g y_g + p_z y_z - \frac{1}{v^2} p_t y_t) = \frac{1}{v^2 t\beta^3}(\beta^3 + z(vt)^2) - \frac{z(g-s)^2}{v^2 t\beta^3} = \frac{2d}{v^2 t\beta} =$$

$$= \frac{(1+r)^2}{2vd} [\ 1 - (pv)^2\ ]^{1/2} \qquad (2\text{-}9e)$$

Coefficient of Q_{yr} :

$$2(y_g r_g + y_z r_z - \frac{1}{v^2} y_t r_t) = \frac{g-s}{\beta^3} (\beta+z) = \frac{(1+r)^2}{2d}[1 - (pv)^2]^{-1/2} \qquad (2\text{-}9f)$$

<u>Linear velocity model:</u>

Now let us assume that the velocity varies linearly with depth:

$$v(z) = v_0 + bz$$

so that, if we define $d' = 2rd/(1+r)$, then

$$v_d \equiv v(d) = v_0 + b\ d \qquad (3\text{-}2)$$

and

$$v_{d'} \equiv v(d') = v_0 + 2rbd/(1+r) \qquad (3\text{-}3)$$

In this case the integrals for $u(p,d)$ and $\tau(p,d)$((1-5) and (1-6)), after making $v^2 = t$ and $dz = \frac{dv}{dv/dz} = \frac{1}{b} dv$, yield:

$$u(p,d) = p \int_0^d \frac{1}{v}[1-(pv)^2]^{-1/2} dz = \frac{1}{pb} \{[1-(pv_0)^2]^{1/2} - [1-(pv_d)^2]^{1/2}\} \qquad (3\text{-}4)$$

and

$$\tau(p,d) = \int_0^d \frac{1}{v}[1-(pv)^2]^{-1/2} dz = \frac{1}{2b} \ln \frac{\{[1-(pv_d)^2]^{1/2}-1\}\cdot\{[1-(pv_0)^2]^{1/2}+1\}}{\{[1-(pv_d)^2]^{1/2}+1\}\cdot\{[1-(pv_0)^2]^{1/2}-1\}}, \quad (3\text{-}5)$$

therefore, transformation (1-1) becomes:

$$s = y + \frac{1}{pb}\{[1-(pv_d)^2]^{1/2} - [1-(pv_0)^2]^{1/2}\} \quad (3\text{-}6a)$$

$$z = 2\ rd\ /\ (1+r) \quad (3\text{-}6b)$$

$$g = y - \frac{1}{pb}\{[1-(pv_d)^2]^{1/2} - [1-(pv_{d'})^2]^{1/2}\} \quad (3\text{-}6c)$$

$$t = \frac{1}{b}\ln \frac{\{[1-(pv_d)^2]^{1/2}-1\}\cdot\{[1-(pv_{d'})^2]^{1/2}+1\}^{1/2}\cdot\{[1-(pv_0)^2]^{1/2}+1\}^{1/2}}{\{[1-(pv_d)^2]^{1/2}+1\}\cdot\{[1-(pv_{d'})^2]^{1/2}-1\}^{1/2}\cdot\{[1-(pv_0)^2]^{1/2}-1\}^{1/2}} \quad (3\text{-}6d)$$

Although this transformation looks somehow complex (especially after remembering (3-2) and (3-3)), it comes out that it has a very simple Jacobian. Following the same procedure we used to compute u and τ, we easily get for the integrals (1-13):

$$I_1(p,d) = \int_0^d v[1-(pv)^2]^{-1/2}\, dz = \frac{1}{bp^2}\{[1-(pv_0)^2]^{1/2} - [1-(pv_d)^2]^{1/2}\} \quad (3\text{-}7a)$$

$$I_2(p,d) = \int_{d'}^d v[1-(pv)^2]^{-1/2} dz = \frac{1}{bp^2}\{[1-(pv_{d'})^2]^{1/2} - [1-(pv_d)^2]^{1/2}\} \quad (3\text{-}7b)$$

$$I_3(p,d) = \int_0^d v^3[1-(pv)^2]^{-3/2} dz = \frac{1}{bp^4}\{[1-(pv_d)^2]^{1/2} + [1-(pv_d)^2]^{-1/2} -$$

$$- [1-(pv_0)^2]^{1/2} - [1-pv_0)^2]^{-1/2}\} \quad (3\text{-}7c)$$

$$I_4(p,d) = \int_{d'}^{d} v^3[1-(pv)^2]^{-3/2}dz = \frac{1}{bp^4}\{[1-(pv_d)^2]^{1/2}+[1-(pv_d)^2]^{-1/2} -$$

$$- [1-(pv_{d'})^2]^{1/2}-[1-(pv_{d'})^2]^{-1/2}\} \qquad (3\text{-}7d)$$

$$I_5(p,d) = \int_{0}^{d'} v[1-(pv)^2]^{-3/2}dz = \frac{1}{bp^2}\{[1-(pv_{d'})^2]^{-1/2}- [1-(pv_0)^2]^{-1/2}\} \quad (3\text{-}7e)$$

$$I_6(p,d) = \int_{d'}^{d} v[1-(pv)^2]^{-3/2}dz = \frac{1}{bp^2}\{[1-(pv_d)^2]^{-1/2}- [1-(pv_{d'})^2]^{-1/2}\} \quad (3\text{-}7f)$$

By using these values and substituting them into (1-14) through (1-17), or by direct differentiation of (3-6), we get for the corresponding Jacobian:

$$s_{p;y;d;r} = \frac{1}{bp^2}\{[1-(pv_0)^2]^{-1/2}- [1-(pv_d)^2]^{-1/2}\} \; ; \; 1 \; ; \; -pv_d[1-(pv_d)^2]^{-1/2} \; ; \; 0 \qquad (3\text{-}8a)$$

$$z_{p;y;d;r} = 0 \; ; \; 0 \; ; \; 2r/(1+r) \; ; \; 2d/(1+r)^2 \qquad (3\text{-}8b)$$

$$g_{p;y;d;r} = \frac{1}{bp^2}\{[1-(pv_d)^2]^{-1/2}- [1-(pv_{d'})^2]^{-1/2}\}; \; 1 \; ; \; pv_d[1-(pv_d)^2]^{-1/2} -$$

$$- \frac{2r}{1+r}pv_{d'}[1-(pv_{d'})^2]^{-1/2}; \; - \frac{2d}{(1+r)^2}pv_{d'}[1-(pv_{d'})^2]^{-1/2} \; . \qquad (3\text{-}8c)$$

$$t_{p;y;d;r} = \frac{1}{bp}\{2[1-(pv_d)^2]^{-1/2}-[1-(pv_{d'})^2]^{-1/2}-[1-(pv_0)^2]^{-1/2}\}; \; 0 \; ; \qquad (3\text{-}8d)$$

$$; \; \frac{2}{v_d}[1-(pv_d)^2]^{-1/2}- \frac{2r}{1+r}\frac{1}{v_{d'}}[1-(pv_{d'})^2]^{-1/2}; - \frac{2d}{(1+r)^2}\frac{1}{v_{d'}}[1-(pv_{d'})^2]^{-1/2}.$$

Since I was interested in studying this model mainly in order to test computer algorithms, I didn't go farther into calculating analytically the inverse of this Jacobian and, therefore, the equation's coefficients.

A last word could be said in relation to the expressions obtained for this model. Since, in most of the expressions, and particularly in the integrals (3-7) we have b in the denominator, we could expect trouble when going from the linear to the constant model as $b \to 0$. Nevertheless, it can be shown that all the expressions tend to the corresponding ones in the constant velocity model as $b \to 0$. We will show this for the case of I_1 :

$$I_1 = \frac{1}{bp^2}\{[1-(pv_0)^2]^{1/2} - [1-(pv_d)^2]^{1/2}\} = \frac{1}{p^2}\frac{D}{v_d - v_0}\frac{p^2(v_d^2 - v_0^2)}{[1-(pv_0)^2]^{1/2}+[1-(pv_d)^2]^{1/2}}$$

$$= D\frac{v_0 + v_d}{[1-pv_0)^2]^{1/2}+[1-(pv_d)^2]^{1/2}} \xrightarrow[v_d \to v_0]{} Dv[1-(pv_0)^2]^{-1/2} , \qquad (3\text{-}9)$$

which is the exact expression for I_1 in the constant velocity model.

Computer algorithms for the general case:

Although the computation of the Jacobian (1-8) as well as its inverse seems to be a straightforward and simple computational problem, nevertheless, a word of caution has to be said in relation to the corresponding computer algorithms.

Notice that the main part of the computation is connected with the evaluation of the integrals (1-13). Since all of them have a singularity at $pv = 1$, we may encounter troubles with its evaluation for values of pv close to 1 .

Here we will discuss a method of evaluation, proposed by Bob Anderssen, which avoids or at least softens this difficulty by integrating the singularity analytically. It is based on the numerical technique of product integration, first discussed in detail by Young (Proc. Roy. Soc., 1954). For the present problem, the proposed method is very simple:

since all of these integrals have analytical solutions for a linearly dependent velocity, we will assume that the velocity varies linearly within each layer, so that in each of these regions we can compute them analytically. Further notice that, as pointed out before, we are dealing with only three different integrals:

$$I_1 = \int_0^d v[\ 1 - (pv)^2\ ]^{-1/2}\ dz \tag{4-1}$$

$$I_2 = \int_0^d v^3[\ 1 - (pv)^2\ ]^{-3/2}\ dz \tag{4-2}$$

$$I_3 = \int_0^d v[\ 1 - (pv)^2\ ]^{-3/2}\ dz \quad . \tag{4-3}$$

Then, following the same procedure used in the previous part to compute these integrals, and assuming:

$$v_i = v_{i-1} + b_i z \quad , \tag{4-4}$$

we get for the first one:

$$I_1 = \frac{1}{p^2} \sum_{j=0}^{Nz-1} \frac{1}{b_j} \{[1 - (pv_j)^2]^{1/2} - [1 - (pv_{j+1})^2]^{1/2}\} \tag{4-5a}$$

Noticing that $b_j = (v_{j+1} - v_j)/Dz$, (for $Dz_1 = Dz_2 = \ldots = Dz$), after repeating the calculations we did in (3-9), we finally get:

$$I_1 = Dz \sum_{j=0}^{Nz-1} \frac{v_j + v_{j+1}}{[1-(pv_j)^2]^{1/2} + [1-(pv_{j+1})^2]^{1/2}} \tag{4-5b}$$

For the second integral we then have:

$$I_2 = \frac{1}{p^4} \sum_{j=0}^{Nz-1} \frac{1}{b_j} \{[1-(pv_{j+1})^2]^{1/2} + [1-(pv_{j+1})^2]^{-1/2} - [1-(pv_j)^2]^{1/2} - [1-(pv_j)^2]^{-1/2}\}, \tag{4-6a}$$

which after some algebra becomes:

$$I_2 = \frac{Dz}{p^2} \sum_{j=0}^{Nz-1} (v_j+v_{j+1}) \frac{1 - [1-(pv_j)^2]^{1/2} [1-(pv_{j+1})^2]^{1/2}}{[1-(pv_j)^2]^{1/2}[1-(pv_{j+1})^2]^{1/2}\{[1-(pv_j)^2]^{1/2}+[1-(pv_{j+1})^2]^{1/2}\}} \tag{4-6b}$$

The last integral comes out to be

$$I_3 = \frac{1}{p^2} \sum_{j=0}^{Nz-1} \frac{1}{b_j} \{[1-(pv_{j+1})^2]^{-1/2} - [1-(pv_j)^2]^{-1/2} \} \tag{4-7a}$$

or

$$I_3 = Dz \sum_{j=0}^{Nz-1} \frac{v_j + v_{j+1}}{[1-(pv_j)^2]^{1/2}[1-(pv_{j+1})^2]^{1/2}\{[1-(pv_j)^2]^{1/2}+[1-(pv_{j+1})^2]^{1/2}\}} \tag{4-7b}$$

Notice that all these three integrals give the correct values for the v = const case when we set $v_1=v_2=\ldots v_{j\ldots}=v$. We could use both expressions (4-5a), (4-6a), (4-7a) or (4-5b), (4-6b), (4-7b) for evaluating the integrals depending on the specific problem we will be solving: If we expect b_i to be zero for some values, we rather shall use expressions "b" . On the other hand, if we are sure that b_i is always different from zero, but we are forced to work near the singularity, we rather shall use expressions "a" . This last case will be discussed in more detail in the next paper, where the "h-frame" for layered media will be analyzed. If both situations hold, instead of "a" or "b" we shall use the already obtained expressions for a constant velocity model.