The Emergent Angle Frame

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The appearance of the directional derivative $(\partial_d + k/d\ \partial_k)$ in the Claerbout-Doherty 1972 paper and in the final result of "A Wide Offset Migration Equation" suggested that a coordinate frame with a component along the direction of $(\partial_d + k/d\ \partial_k)$ might have an even simpler equation for migration. This turns out to be true and easy to verify.

For simplicity we set the velocity v equal unity. (To convert the final results to an arbitrary velocity it is only necessary to replace t by $v\,t$.) The frame of interest is

$$p = x / t \tag{1a}$$

$$r = 2z / ((t^2 - x^2)^{1/2} + z) \tag{1b}$$

$$d = ((t^2 - x^2)^{1/2} + z) / 2 \tag{1c}$$

and its inverse is

$$x = (2-r)\ p\,d / (1-p^2)^{1/2} \tag{2a}$$

$$z = r\,d \tag{2b}$$

$$t = (2-r)\ d / (1-p^2)^{1/2} \tag{2c}$$

Figure 1 gives a geometrical view.

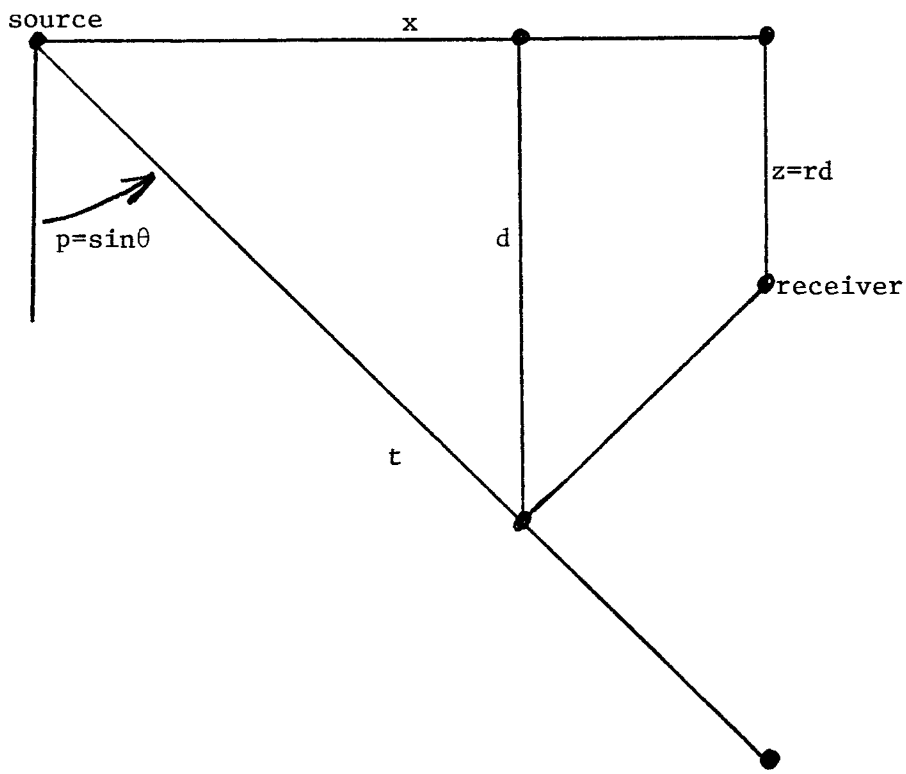

Figure 1. The geometry of the transformation. A complete set of descriptors is either (p, r, d) or (x, z, t) .

We'll need all the derivatives of (p, r, d). Using s as an abbreviation for $(t^2 - x^2)^{1/2} = (2-r)\, d$ we get

$$p_x = 1 / t \tag{3a}$$

$$p_z = 0 \tag{3b}$$

$$p_t = - x / t^2 \tag{3c}$$

$$r_x = 2zx / (s(s+z)^2) \tag{3d}$$

$$r_z = 2s / (s+z)^2 \tag{3e}$$

$$r_t = -2zt / (s(s+z)^2) \tag{3f}$$

$$d_x = -x/2s \tag{3g}$$

$$d_z = 1/2 \tag{3h}$$

$$d_t = t/2s \tag{3i}$$

These will be inserted into the wave equation in the form

$$[(p_x \partial_p + r_x \partial_r + d_x \partial_d)^2 + (p_z \partial_p + r_z \partial_r + d_z \partial_d)^2 - (p_t \partial_p + r_t \partial_r + d_t \partial_d)^2] Q = 0 \tag{4}$$

In (4) the Q_{dd} term is

$$Q_{dd} (d_x^2 + d_z^2 - d_t^2) = 0 \tag{5}$$

The Q_{dp} term is

$$Q_{dp} \, 2 (p_x d_x + p_z d_z - p_t d_t) = 0 \tag{6}$$

The Q_{rp} term is

$$Q_{rp} \, 2 (p_x r_x + p_z r_z - p_t r_t) = 0 \tag{7}$$

We call the above three terms convection terms. In a homogeneous material convection should be done by the transformation itself, not by the partial differential equation. Now let's get on to the non-zero terms. The Q_{rd} term is

$$Q_{rd}\ 2\,(r_x d_x + r_z d_z - r_t d_t) =$$

$$= 1 \,/\, ((t^2 - x^2)^{1/2} + z) = Q_{rd} \,/\, d \qquad (8)$$

The Q_{pp} term is

$$Q_{pp}\,(p_x{}^2 + p_z{}^2 - p_t{}^2) = 1/t^2 - x^2 / t^4 = \frac{(1-p^2)^2}{(2-r)^2 d^2}\, Q_{pp} \qquad (9)$$

The Q_{rr} term is

$$Q_{rr}\,(r_x{}^2 + r_z{}^2 - r_t{}^2) = \frac{(1-r)}{4\,d^2}\, Q_{rr} \qquad (10)$$

The resulting equation including the Fresnel-like approximation is

$$Q_{rd} = \frac{-(1-p^2)^2}{(2-r)^2\, d}\; Q_{pp} \qquad (11)$$

The coefficient in (11) blows up at $d=0$ because Q is such a smooth function of p in the vicinity of $d=0$. This causes no fundamental problem but may be an annoyance when you wish to compute with the explicit method. If (k, r, d) coordinates are preferred over (p, r, d) coordinates for practical reasons, then the question arises whether the Fresnel-like approximation is better in (p, r, d) than in (k, r, d) . To check this I applied the transformation

$$r' = r \qquad (12a)$$

$$d' = d \qquad (12b)$$

$$k = d \tan\theta = p\, d \,/\, (1-p^2)^{1/2} \qquad (12c)$$

to equation (11). Somewhat to my surprise I got the identical equation as found in "A wide offset migration equation." In retrospect, the result is more easily seen by noting that (12c) is independent of r so that $Q_{rr} = Q'_{r'r'}$.

The next two things we'll want to do are go to section coordinates and go to stratified media.