Shifting Frames

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The problem is to try to find a suitable coordinate frame to describe downward continuation of a beam of waves which surround some particular ray. In early work this ray was propagating vertically. Then in the slant frame work the ray had some fixed angle θ to the vertical. Now we are talking about a ray which takes a path through some arbitrary stratified medium. The transformations of the present study are of the general type

$$t' = t + f(x) + g(z) \tag{1a}$$

$$x' = x + h(t) \tag{1b}$$

$$z' = z \tag{1c}$$

Setting

$$P(x, z, t) = Q(x', z', t') \tag{2}$$

The chain rule gives us the usual first derivative equations

$$P_t = Q_{t'} t'_t + Q_{x'} x'_t + Q_{z'} z'_t$$

$$= (\partial_{t'} + x'_t \partial_{x'}) Q \tag{3a}$$

$$P_x = Q_{t'} t'_x + Q_{x'} x'_x + Q_{z'} z'_x$$

$$= (\partial_{x'} + t'_x \partial_{t'}) Q \tag{3b}$$

$$P_z = Q_{t'} t'_z + Q_{x'} x'_z + Q_z z'_z$$

$$= (\partial_{z'} + t'_z \partial_{t'}) Q \tag{3c}$$

The usual assumption that gradients of the waves will sufficiently exceed gradients of the coordinate frame enables us to get second difference operators just by squaring the operators in (3). Inserting these into the wave equation $P_{xx} + P_{zz} = P_{tt} / \tilde{v}^2$ and, for convenience, omitting primes on subscripts of Q we have

$$Q_{xx} + 2t'_x Q_{xt} + (t'_x)^2 Q_{tt} + Q_{zz} + 2t'_z Q_{zt} + (t'_z)^2 Q_{tt}$$

$$= \frac{1}{\tilde{v}^2} \left((x'_t)^2 Q_{xx} + 2x'_t Q_{xt} + Q_{tt} \right) \tag{4}$$

We often like to have the coefficient of Q_{tt} vanish. To see why, suppose Q is independent of x, neglect Q_{zz} and note (4) is of the form $(Q_z + \text{(non-zero)}\ Q_t = 0)_z$. Such an equation is merely a translator and we usually feel that translation should be done by the coordinate transformation, not by the differential equation. The vanishing of the coefficient of Q_{tt} is expressed by the condition

$$(t'_x)^2 + (t'_z)^2 = 1 / \tilde{v}^2 \tag{5}$$

Equation (5) is called the eikonal equation and it has a simple physical meaning. Let $t'(x,z)$ be the arrival time of a wave in a region of the (x,z) plane. How long does it take for the wave front to go one unit of distance? Distance is velocity times time and we have vector quantities, so

$$\text{(total distance)}^2 = \text{(horiz. distance)}^2 + \text{(vert. distance)}^2$$

or

$$1^2 = (\tilde{v}\ t'_x)^2 + (\tilde{v}\ t'_z)^2$$

which is essentially (5).

One way we can assure the condition (5) is to take the t' transformation to be given by

$$t' = t \pm \int_{z_o}^{z} \frac{\cos \bar{\theta}(z)}{\bar{v}(z)} \, dz - x \frac{\sin \bar{\theta}(z)}{\bar{v}(z)} \tag{6}$$

Where a ray going through a stratified medium with velocity $\bar{v}(z)$ has an angle from the vertical of $\theta(z)$. In order for (6) to assure the exact satisfaction of (5) it will be necessary that the velocity in the wave equation $\tilde{v}$ agree with $\bar{v}$. (However the exact extinguishing of the Q_{tt} term is neither necessary nor possible if $\tilde{v} = \tilde{v}(x,z)$.) Also, in checking that the form of (6) does indeed assure satisfaction of (5) it is helpful to recall Snell's law which says that $\sin \bar{\theta}(z) / \bar{v}(z)$ is a constant independent of z for any ray path.

We still have not nailed down the x' transformation. This can be done by eliminating the other translation type term in the differential equation (4), namely Q_{xt} . Setting its coefficient to zero gives

$$t'_x = x'_t / \tilde{v}^2 \tag{7}$$

A transformation x' intended to satisfy (7) is

$$x' = x - \int_{t_o}^{t} \bar{\bar{v}}(t) \sin \bar{\bar{\theta}}(t) \; dt \tag{8}$$

In (8) we have defined $\bar{\bar{v}}$ and $\bar{\bar{\theta}}$ as functions of time. These are the velocity and angle of the coordinate system defining ray as functions of time. Clearly by substitution, if $\tilde{v} = \bar{v} = \bar{\bar{v}}$ then (7) is satisfied. With the transformation (1) now fully defined, the wave equation (4) in the new coordinate frame, with the Fresnel-like approximation $Q_{zz} = 0$ in stratified media $\tilde{v} = \bar{v}(z)$ is

$$Q_{xx} + 2\ t'_z\ Q_{zt} = \bar{v}^{-2}\ (x'_t)^2\ Q_{xx}$$

$$2\ \frac{\cos\ \bar{\theta}}{\bar{v}}\ Q_{zt} = (\ \sin^2\theta - 1\ )\ Q_{xx}$$

$$Q_{zt} = \pm\ .5\ \bar{v}(z)\ \cos(\theta(z))\ Q_{xx} \qquad (9)$$